Simultaneous Diophantine approximations and generalizations of the continued fraction

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Abstract

In the space $\mathbb{R}^n$, suppose that we are given l homogeneous linear forms and k homogeneous quadratic forms; each quadratic form is the product of two complex conjugate linear forms, and $l + 2k = n$. The moduli of all m forms $(m = k + l)$ define a mapping of $\mathbb{R}^n$ to the nonnegative orthant $\mathbb{R}^m_+$ of the space $\mathbb{R}^m$. Nonzero integer points from $\mathbb{R}^n$ are mapped to a set $\mathbf{Z} \subset \mathbb{R}^m_+$. The closure of the convex hull $\mathbf{G}$ of $\mathbf{Z}$ is a polyhedral set in $\mathbb{R}^m_+$. Its boundary $\partial\mathbf{G}$ is of dimension $m - 1$ and contains the images of the best Diophantine approximations to the root subspaces of all m forms. In the algebraic case, m forms are related in a certain manner to the roots of an irreducible polynomial of degree n that has l real roots and k pairs of complex conjugate roots. It is proved that, in the algebraic case, the boundary $\partial\mathbf{G}$ has $m - 1$ independent periods. This is a generalization of Lagrange's theorem on the periodicity of the continued fraction of a quadratic irrationality.

1. Statement of the problem

Let the linear form $l_1(X) = \langle J, X \rangle \stackrel{\text{def}}{=} j_1 x_1 + j_2 x_2 + j_3 x_3$ and the quadratic form $l_2(X) = \langle K, X \rangle \langle \overline{K}, X \rangle$ be given in $\mathbb{R}^3$ with coordinates $X = (x_1, x_2, x_3)$. The product is $f(X) = l_1(X) l_2(X)$. The root set $f(X) = 0$ is $\mathcal{L}_1 \cup \mathcal{L}_2$, where the plane $\mathcal{L}_1 = \{X : \langle J, X \rangle = 0\}$, the line $\mathcal{L}_2 = \{X : \langle \mathrm{Re}\, K, X \rangle = \langle \mathrm{Im}\, K, X \rangle = 0\}$. We assign

$$m_i(X) = |l_i(X)|, \quad i = 1, 2; \qquad M(X) = (m_1(X), m_2(X)).$$

An integer point $X \in \mathbb{Z}^3$ is one of *the best approximations to* $\mathcal{L}_1 \cup \mathcal{L}_2$ if there is no point $Y \in \mathbb{Z}^3$, $Y \neq 0$ such that

$$M(Y) \leq M(X), \qquad \|M(Y)\| < |M(X)\|.$$

Problem. *Find an algorithm for computation of best approximations.*

Algorithms of solution of the Problem

Now there are two algorithms solving the problem: by G. Voronoi (1896), it tends to $\mathcal{L}_1$, and by A. Bruno and V. Parusnikov (2005), it tends to $\mathcal{L}_2$. Here we propose a new algorithm working in both directions: to $\mathcal{L}_1$ and to $\mathcal{L}_2$.

2. The principal construction

The vector-function

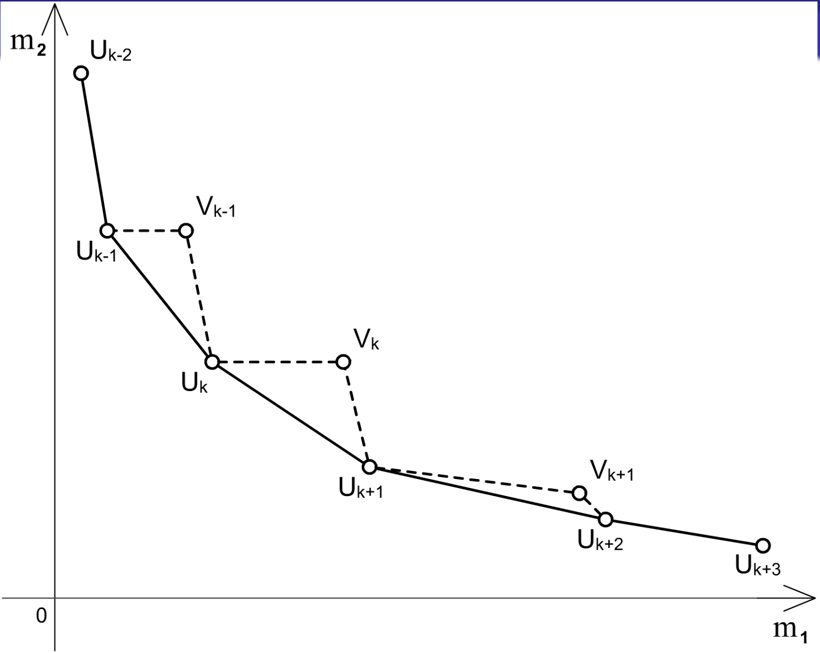
$$M(X) = (m_1(X), m_2(X)) = (|l_1(X)|, |l_2(X)|)$$

maps $\mathbb{R}^3$ into the first quadrant S_+ of the plane $S = \mathbb{R}^2 \ni (m_1, m_2)$. Let $\mathbf{Z}^2$ be the image of $\mathbb{Z}^3$ except $X = 0$:

$$\mathbf{Z}^2 = M(\mathbb{Z}^3 \setminus 0).$$

$\mathbf{M}$ is the convex hull of $\mathbf{Z}^2$, and $\partial\mathbf{M}$ is the boundary of $\mathbf{M}$. $\partial\mathbf{M}$ is the convex open polygon (Fig. 1); it consists of vertices and edges. All its vertices are images of the integer points $X \in \mathbb{Z}^3$. Some of the images can be on the edges. All points of $\partial\mathbf{M} \cap \mathbf{Z}^2$ are images of the best approximations. So computing the polygon $\partial\mathbf{M}$ is sufficient for solving the Problem.

Fig. 1



The function ζ

For two points $U = (u_1, u_2)$ and $V = (v_1, v_2) \in S_+$ the function

$$\zeta_1(U, V) = \frac{v_2 - u_2}{u_2(u_1 - v_1)},$$

is the value m_1^{-1} at the point $(m_1, 0) \in S_+$ of the axis m_1 where the axis intersects the line going through points U and V.

3. The Algorithm

Assume that we have the basis $B_1,\ B_2,\ B_3 \in \mathbb{Z}^3$, $\det(B_1^*, B_2^*, B_3^*) = \pm 1$, ordered in some manner. We compute the next basis $\tilde{B}_1,\ \tilde{B}_2,\ \tilde{B}_3$ by the linear transformation

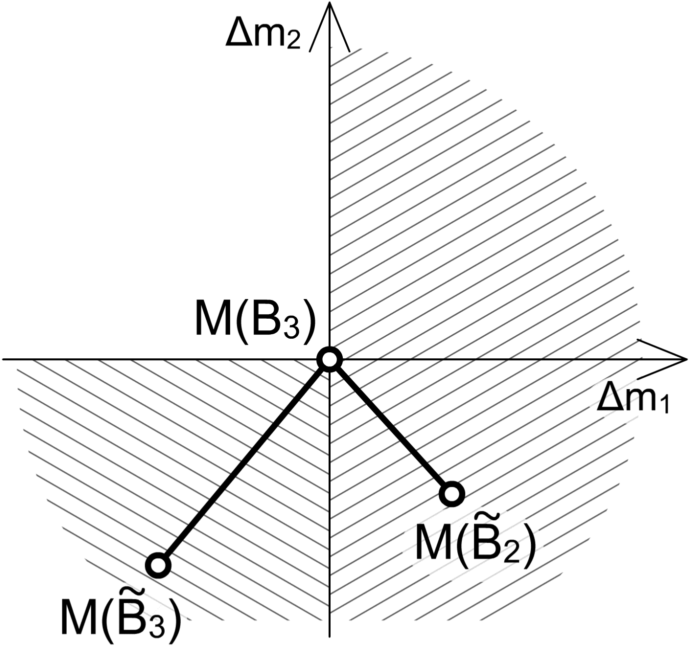
$$\begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & a \\ 1 & b & c \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},$$

where $a,\ b,\ c \in \mathbb{Z}$; $|a|,\ |b|,\ |c| \le \kappa_k$, differences $M(\tilde{B}_2) - M(B_3)$, $M(\tilde{B}_3) - M(B_3)$ belong either to the quadrant III or to the quadrants I and IV (Fig. 2);

$$\tilde{B}_2 :\ \zeta_1(M(B_3), M(\tilde{B}_2)) = \max_a \zeta_1(M(B_3), M(\tilde{B}_2));$$

$$\tilde{B}_3 :\ \zeta_1(M(B_3), M(\tilde{B}_3)) = \max_{b,c} \zeta_1(M(B_3), M(\tilde{B}_3)).$$

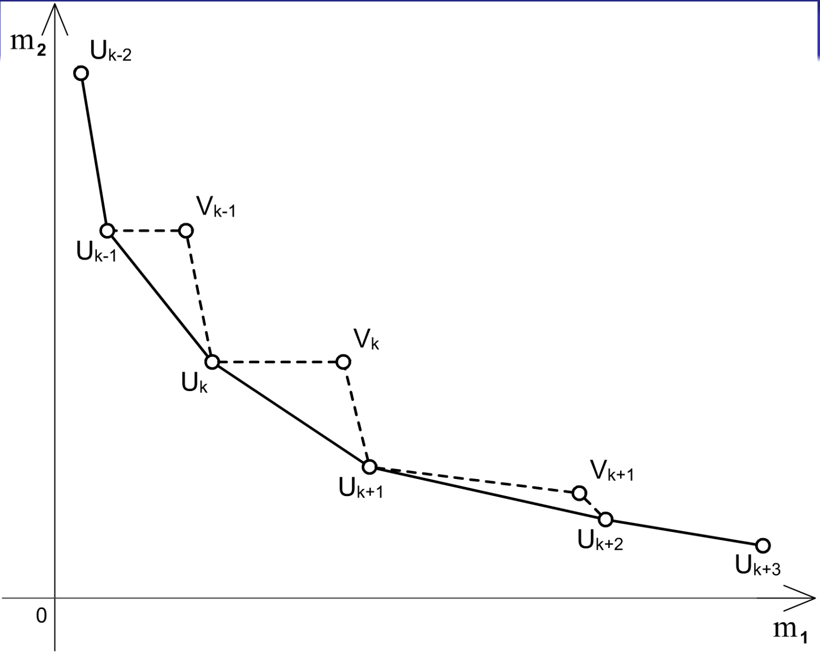
Fig. 2



The Algorithm

Among all possible values of $\tilde{B}_2$ and $\tilde{B}_3$, we choose those that give the greatest inclinations for lines going through points $M(\tilde{B}_2)$, $M(\tilde{B}_3)$, and the point $M(B_3)$.
Here $\kappa_k = 3 \cdot 2^k$ is the movable boundary for $|a|$, $|b|$, $|c|$: if one of $|a|$, $|b|$, $|c|$ reaches κ_k, then we replace κ_k by κ_{k+1}, and repeat computation of $\tilde{B}_2$ and $\tilde{B}_3$. To the basis $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$, we again apply the described algorithm and so on (Fig. 1). The vectors $\tilde{B}_i$ of the basis tend to $\mathcal{L}_2$ in this algorithm. To obtain $\tilde{B}_i \to \mathcal{L}_1$ as $i \to \infty$, it is sufficient to permute forms $l_1(X)$ and $l_2(X)$. So the algorithm works in both directions: to $\mathcal{L}_1$ and to $\mathcal{L}_2$.

Fig. 1



4. Algebraic case

Let the polynomial $P(\lambda) \stackrel{\text{def}}{=} \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma$ have integer coefficients and negative discriminant. It has 3 roots $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \overline{\lambda_3} \in \mathbb{C}$. We assign $J = (1, \lambda_1, \lambda_1^2)$, $K = (1, \lambda_2, \lambda_2^2)$, $l_1(X) = \langle J, X \rangle$, $l_2(X) = \langle K, X \rangle \langle \overline{K}, X \rangle$. According to the Dirichlet Theorem, the field $\mathbb{Q}(\lambda_1)$ has one fundamental unity which corresponds to the unimodular substitution $X = DY$. The substitution is the automorphism of $|f(X)| = |l_1(X)||l_2(X)|$ and of the open polygon $\partial\mathbf{M}$. Thus, the polygon $\partial\mathbf{M}$ is periodic.
So our algorithm allows to find the mimimal period of $\partial\mathbf{M}$ and the corresponding fundamental unit of the field.
The algorithm was implemented and verified on a large number of polynomials.

5. Three linear forms and positive discriminant

The similar approach in the case of three linear forms leads to consideration of a polyhedral surface $\partial\mathbf{M}$ in the 3-dimensional first octant $(m_1, m_2, m_3) \geq 0$. In the algebraic case the polynomial $P(\lambda)$ must have positive discriminant; then three real roots of $P(\lambda)$ give 3 linear forms. The surface $\partial\mathbf{M}$ has 2 independent periods corresponding to 2 fundamental units. The algorithm was described recently (2005–2007).

6. General situation

In $\mathbb{R}^n$ there are given ℓ linear forms $f_i(X)$, $i = 1, \ldots, \ell$ and k quadratic forms $f_j(X)$, $j = \ell + 1, \ldots, \ell + k$ and $\ell + 2k = n$. The map

$$m_i = |f_i(X)|, \qquad i = 1, \ldots, \ell + k$$

transforms $\mathbb{Z}^n \setminus 0$ into the set $\mathbf{Z}^{\ell+k} \subset \mathbb{R}_+^{\ell+k}$. $\mathbf{M} \subset \mathbb{R}_+^{\ell+k}$ is the convex hull of $\mathbf{Z}^{\ell+k}$. Boundary $\partial\mathbf{M}$ has dimension $\ell + k - 1$. In algebraic case $\partial\mathbf{M}$ has $\ell + k - 1$ independent periods that correspond to fundamental units. The periods and units can be found by computation of $\partial\mathbf{M}$ by a similar algorithm.

7. Comparison with other approaches

In 1895–1896 F. Klein, H. Minkowski and G. Voronoi proposed 3 different approaches to generalization of the continued fraction for the case of 3 linear forms in $\mathbb{R}^3$. Our approach is nearer to that of Voronoi, but is different from it.

The Klein's approach was proposed again independently by B. Skubenko (1988) and by V. Arnold (1993). The term "Klein's polyhedra" I introduced (1994) as a reaction to the term "Arnold's polyhedra" introduced by G. Lachaud (1993). We found (1994–2002) that Klein's polyhedra cannot give a background for an algorithm generalizing the continued fraction. So I proposed (2003) one polyhedron **M** which is, in a sense, the convex hull of 8 Klein's polyhedra. Nevertheless now several groups in different countries study the Klein's polyhedra.

The Minkowski's approach was developed for n linear forms in $\mathbb{R}^n$ by J. Lagarias (1994). But his algorithm is essentially more complicated than algorithm proposed by the author.

Bibliography

This is a joint work with V. Parusnikov.

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