Multidimensional Gauss Reduction Theory for conjugacy classes of SL(n,Z)

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19 August 2010

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Strategy: **find normal forms**.

Example

In the classical case of algebraically closed field any matrix is conjugate to Jordan normal form. The set of Jordan blocks is the complete invariant of a conjugacy class.

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We study the simplest case: *all eigenvalues are distinct*.

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$SL(2,\mathbb{Z}) \rightarrow$ complete invariant $\rightarrow$ "almost" normal form.

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$SL(2,\mathbb{Z}) \rightarrow$ complete invariant $\rightarrow$ "almost" normal form.

$SL(n,\mathbb{Z})$:

- find complete invariant;
- write an analog of "almost" normal forms;
- study what "almost" mean in this case.

The case of $SL(2, \mathbb{Z})$

- complex case: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
- totally real case: Gauss Reduction Theory
- degenerate case of double roots: $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for $n \geq 0$.

Ordinary continued fractions

The expression (finite or infinite)

$$a_0 + 1/(a_1 + 1/(a_2 + \ldots)\ldots))$$

is an *ordinary continued fraction* if $a_0 \in \mathbb{Z}$, $a_k \in \mathbb{Z}_+$ for $k > 0$.
Denote it $[a_0 : a_1; \ldots]$ (or $[a_0 : a_1; \ldots ; a_n]$).

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Ordinary continued fraction is *odd* (*even*) if it has odd (even) number of elements.

$$\frac{7}{5} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{1}{2 + \frac{1}{1+1/1}}$$

$$\frac{7}{5} = [1 : 2; 2] = [1 : 2; 1; 1]$$

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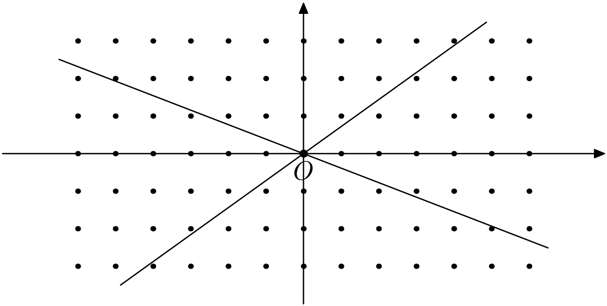
Proposition

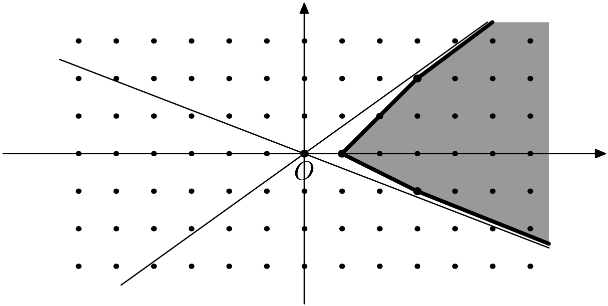
Any rational number has a unique odd and even ordinary continued fractions.
Any irrational number has a unique infinite ordinary continued fraction

The totally real case of $SL(2,\mathbb{Z})$



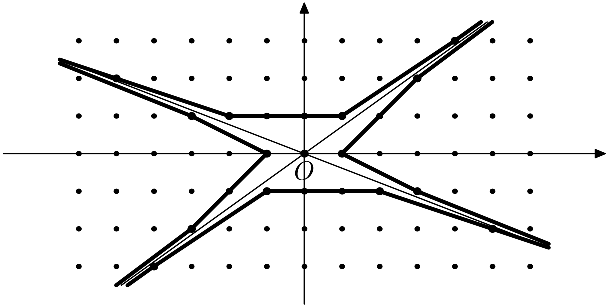
Eigenlines of an operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

The totally real case of $SL(2,\mathbb{Z})$



The *sail* for one of the octants, i.e. the boundary of the convex hull of all integer inner points.

The totally real case of $SL(2, \mathbb{Z})$



The set of all sails is called *geometric continued fraction* (in the sense of Klein).

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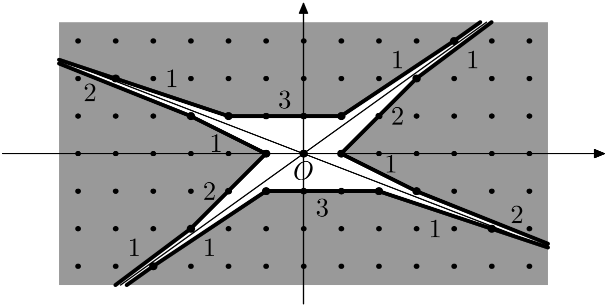

Integer length of a segment is the number of integer inner points in a segment plus one.

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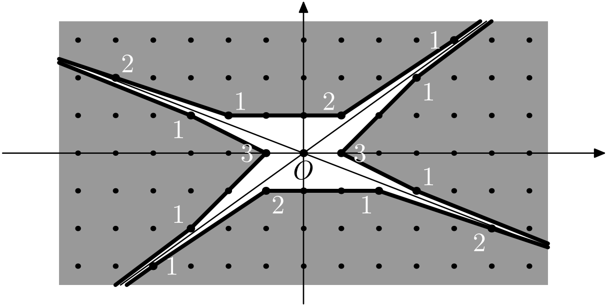

Integer angle is the index of the sublattice generated by points of the edges of the angle in the lattice of integer points.

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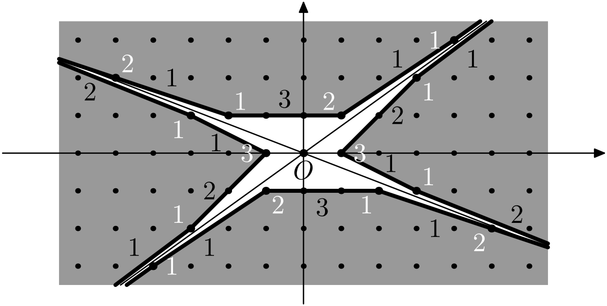

Geometric continued fraction for the operator $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$.

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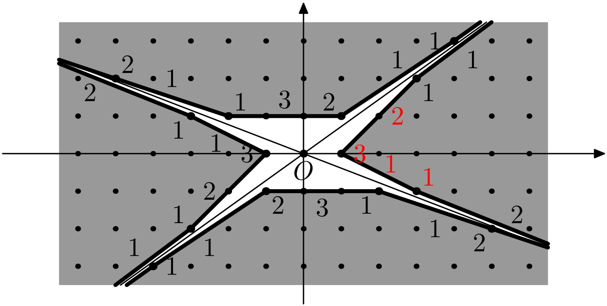
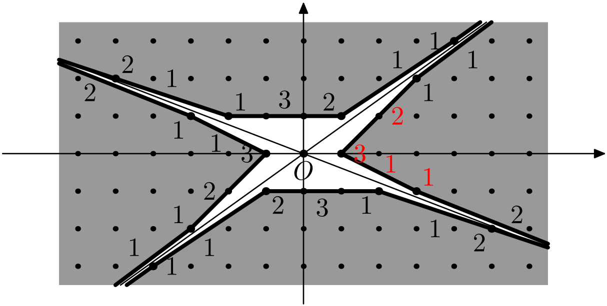


In the case of $SL(2,\mathbb{Z})$ operators the sequences for the sails are periodic.

For instance, for $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$ the period is: $(1,1,3,2)$.

The totally real case of $SL(2,\mathbb{Z})$



Theorem

A period (up to a shift) is a complete invariant of a conjugacy class of an operator in $SL(2,\mathbb{Z})$.

The totally real case of $SL(2,\mathbb{Z})$

Definition

An operator $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is *reduced* if $d > b \geq a \geq 0$.

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Theorem

Suppose $\frac{b}{a} = [a_1; a_2 : \ldots : a_{2n-1}]$ *and* $\lambda = \lfloor \frac{d-1}{b} \rfloor$ *then one of the periods of geometric continued fraction is*

$$(a_1, a_2, \ldots, a_{2n-1}, \lambda).$$

The totally real case of $SL(2,\mathbb{Z})$

Example

For the operator $\begin{pmatrix} 1519 & 1164 \\ -1964 & -1505 \end{pmatrix}$ the period is $(1,2,1,2)$.

Hence a minimal period is $(1,2)$.

The reduced operators conjugate to the given one are: $\begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$ and $\begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix}$.

$SL(n,\mathbb{Z})$ for $n \ge 3$. Notation

Reduced operators $\longrightarrow$ Hessenberg operators

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\
0 & a_{3,2} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
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We say that the matrix M is of *Hessenberg type*

$$\langle a_{1,1}, a_{1,2} | a_{2,1}, a_{2,2}, a_{2,3} | \cdots | a_{n-1,1}, a_{n-1,2}, \ldots, a_{n-1,n} \rangle.$$

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Additionally for any $k, l < k$

$$a_{k+1,k} > a_{l,k} \geq 0.$$

$SL(n, \mathbb{Z})$ for $n \geq 3$. Reduced matrices.

Hessenberg complexity

$$\varsigma(M) := \prod_{j=1}^{n-1} |a_{j+1,j}|^{n-j}$$

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We say that matrix is *reduced* if the complexity is minimal possible in the conjugacy class.

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Theorem

For any $M \in SL(n,\mathbb{Z})$ there exists a reduced Hessenberg matrix H in the conjugacy class of M.

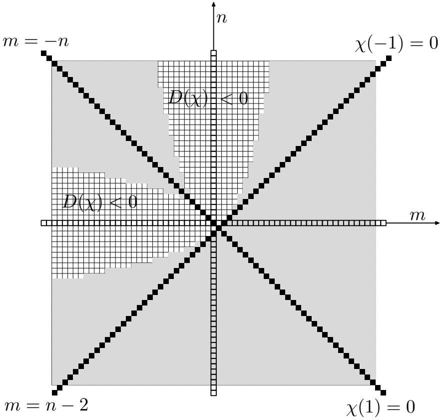
Reduced Hessenberg matrices of $SL(3,\mathbb{Z})$

We will study the simplest case of Hessenberg matrices having 1 real roots and two complex conjugate.

Dark gray boxes on the pictures – non-reduced operators. White boxes – reduced

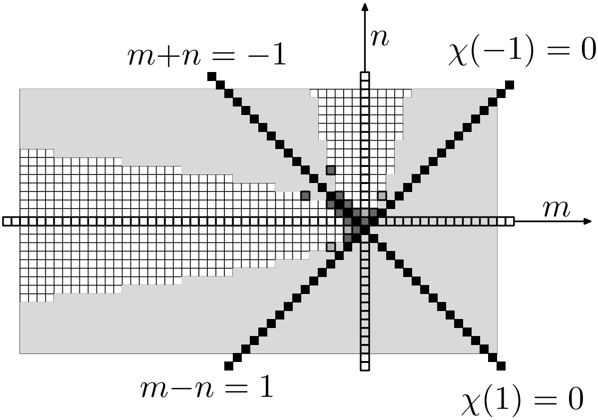
Reduced Hessenberg matrices of $SL(3,\mathbb{Z})$

Hessenberg matrices of type $\langle 0,1|0,0,1\rangle$: $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & m \\ 0 & 1 & n \end{pmatrix}$



Reduced Hessenberg matrices of $SL(3,\mathbb{Z})$

Hessenberg matrices of type $\langle 0,1|1,0,2\rangle$: $\begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 2 & 2n+1 \end{pmatrix}$



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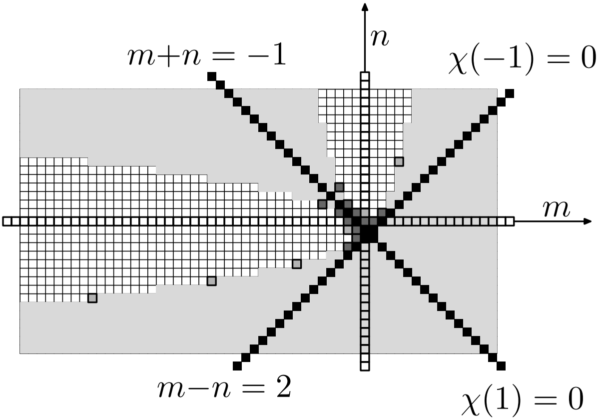

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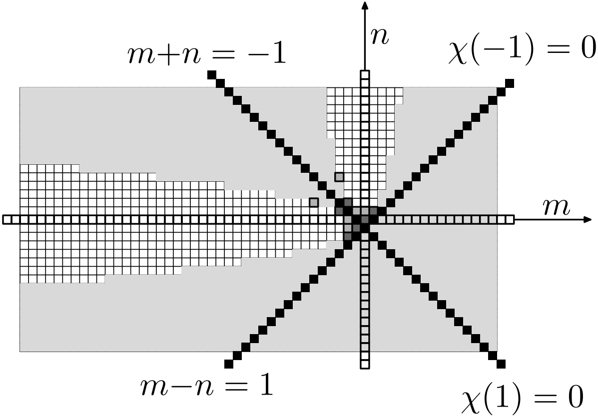
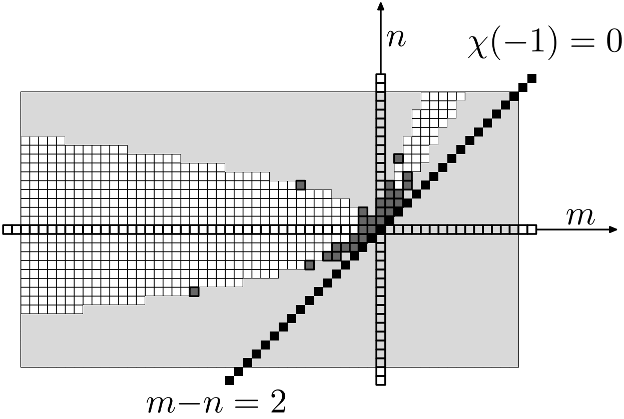
Hessenberg matrices of type $\langle 0, 1 | 1, 0, 3 \rangle$: $\begin{pmatrix} 0 & 1 & n+1 \\ 1 & 0 & m \\ 0 & 3 & 3n+2 \end{pmatrix}$



Reduced Hessenberg matrices of $SL(3,\mathbb{Z})$

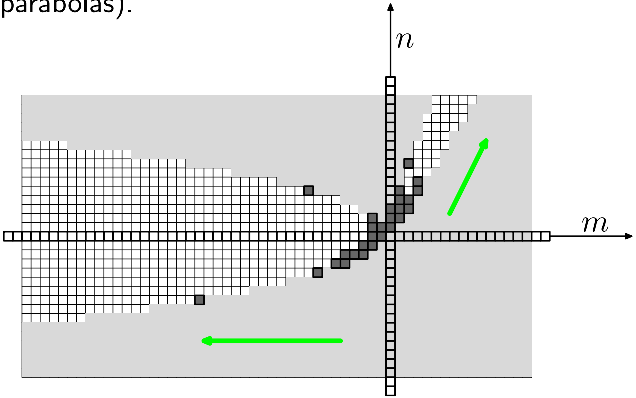
Hessenberg matrices of type $\langle 1,2|1,1,3\rangle$: $\begin{pmatrix} 1 & 1 & 1+m+n \\ 2 & 1 & 2m+n \\ 0 & 3 & 3n-1 \end{pmatrix}$



Main results for $SL(3, \mathbb{Z})$

Remark

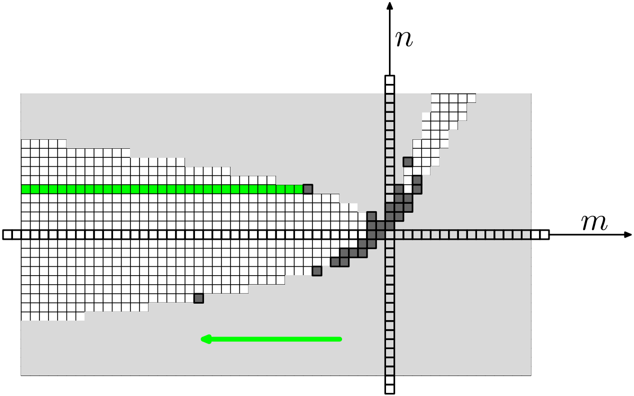
For any Hessenberg type in $SL(3, \mathbb{Z})$ the corresponding family of non-totally real operators has two asymptotic directions (defined by two parabolas).



Main results for $SL(3, \mathbb{Z})$

Theorem

Any ray with asymptotic direction contains a finite number non-reduced operators.



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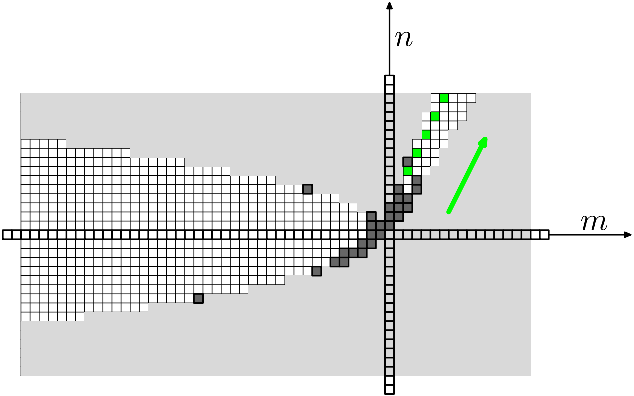
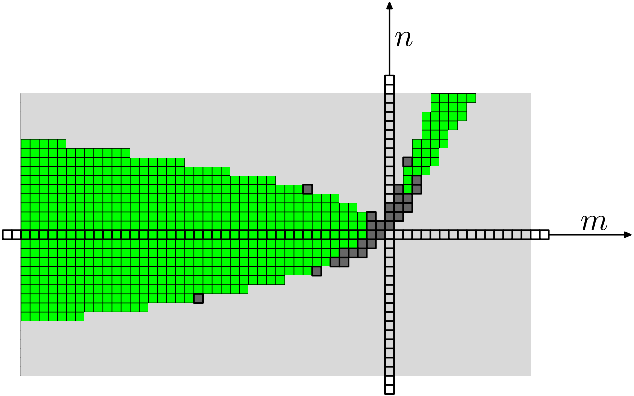


Main results for $SL(3, \mathbb{Z})$

Conjecture

For any Hessenberg type the corresponding family of non-totally real operators contains only finitely many non-reduced operators.



Compare with totally-real case

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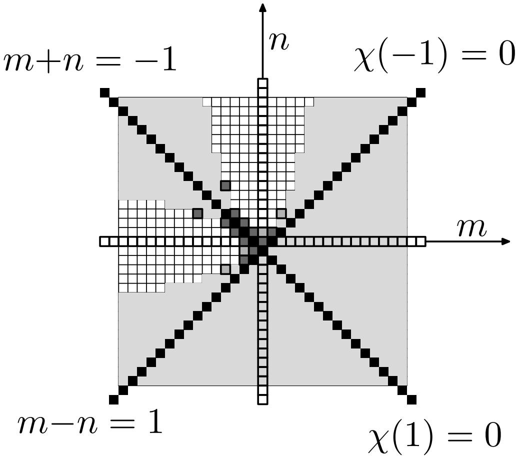
Non-totally-real case.

Hessenberg matrices of type $\langle 0,1|1,0,2\rangle$



Compare with totally-real case

General case.

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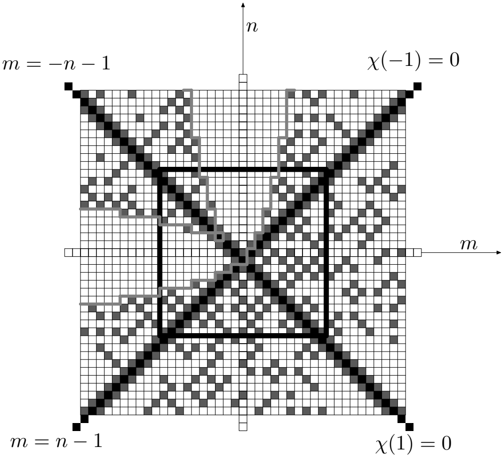
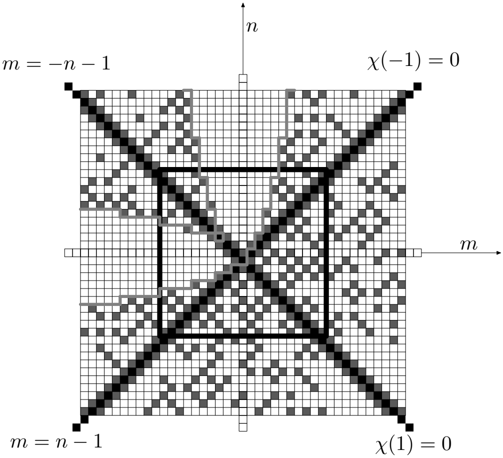


Compare with totally-real case

General case.

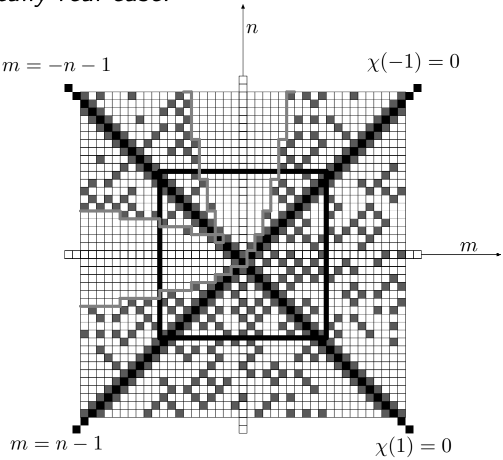
Reduced operators (white) are checked only within a square.



Compare with totally-real case

Problem

Study the totally-real case.



Cases of $SL(4, \mathbb{Z})$

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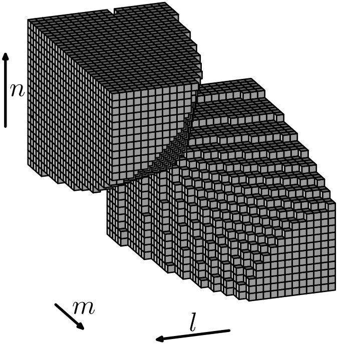
The family of matrices of the Hessenberg type $\langle 0,1|0,0,1|1,3,1,4\rangle$.



a) Two real roots.

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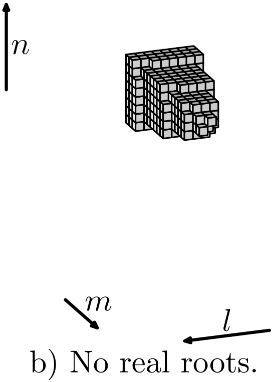
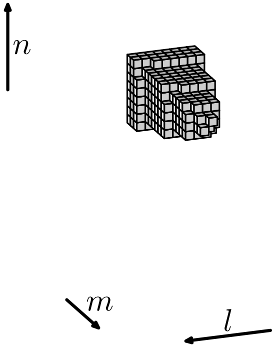


Cases of $SL(4, \mathbb{Z})$

Problem

Study any case of $SL(4, \mathbb{Z})$.



b) No real roots.

Invariants of classes for $SL(3, \mathbb{Z})$

Multidimensional continued fractions is an analog of geometric continued fractions.

Invariants of classes for $SL(3, \mathbb{Z})$

Multidimensional continued fractions is an analog of geometric continued fractions.

F. Klein(1895) – totally real case.

G. Voronoi(1896) – first steps in the rest cases.

J.A. Buchmann(1985) – final definition of Klein-Voronoi continued fraction.

V. Arnold(1989) – geometric approach to continued fractions.

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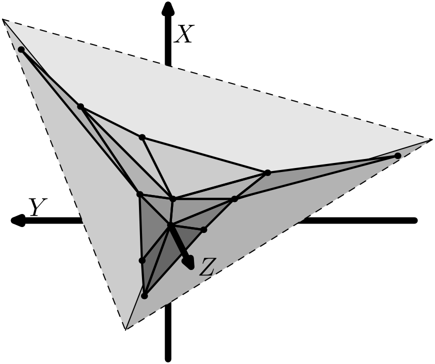
Theorem

A period of the Klein-Voronoi continued fraction is a complete invariant for conjugacy classes of $SL(n,\mathbb{Z})$.

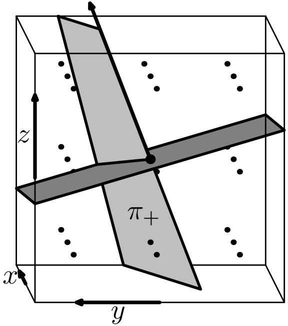
Totally real case

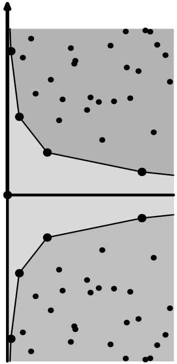


The *sail* for a cone determined by invariant hyperplanes for an operator is the convex hull of all integer inner points of this cone.

The set of all sails is called *geometric continued fraction* (in the sense of Klein).

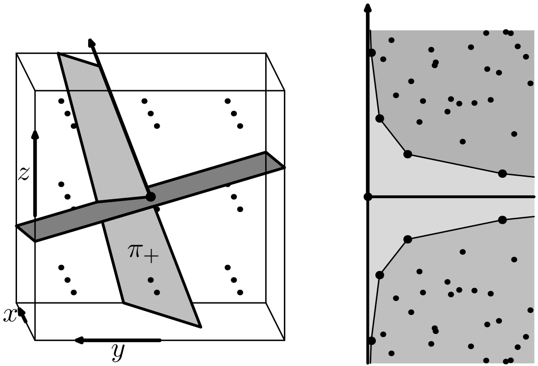
The case of two complex conjugate eigenvectors





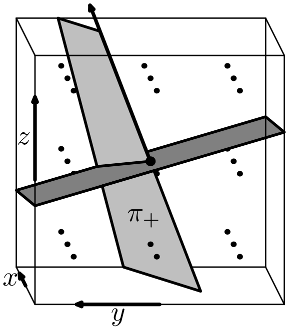
We have one invariant plane and one eigen-line.

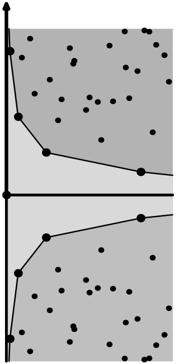
The case of two complex conjugate eigenvectors



The group of $SL(3,\mathbb{R})$ operators commuting with our operator form a circle that defines elliptic fibration of $\mathbb{R}^3$.
Project along the ellipses to π_+.

The case of two complex conjugate eigenvectors

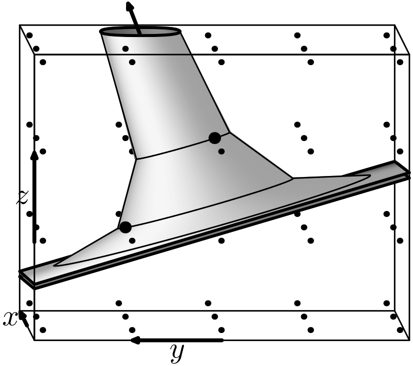




Take the convex hull of all points corresponding to ellipses with an integer point.

The Klein-Voronoi sail is the preimage of the convex hull

The case of two complex conjugate eigenvectors



Take the convex hull of all points corresponding to ellipses with an integer point.

The Klein-Voronoi sail is the preimage of the convex hull

Summary

1. Families of Hessenberg operators describe "very well" conjugacy classes of operators having two complex conjugate eigenvalues.

Theorem

Any ray with asymptotic direction contains a finite number non-reduced operators.

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Theorem

Any ray with asymptotic direction contains a finite number non-reduced operators.

2. A period of a Klein-Voronoi continued fraction is a complete invariant of a conjugacy class. It's characteristics are good to study the structure of the set of all conjugacy classes.

(in particular, we essentially use them in the proofs)

Problems

Conjecture

For any Hessenberg type of $SL(3,\mathbb{Z})$ the corresponding family of non-totally real operators contains only finitely many non-reduced operators.

Problem

Study the totally-real case of $SL(3,\mathbb{Z})$.

Problem

Study any case of $SL(4,\mathbb{Z})$.

Empty frame

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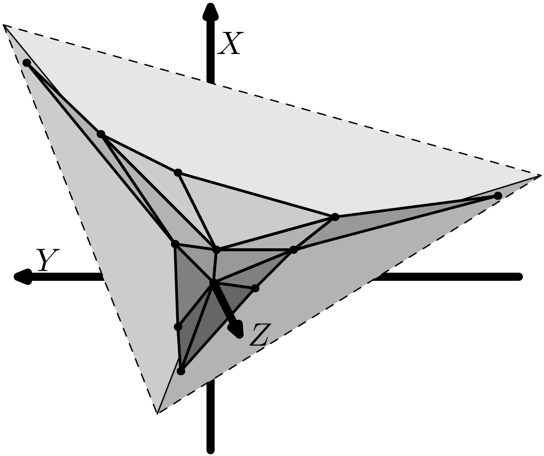
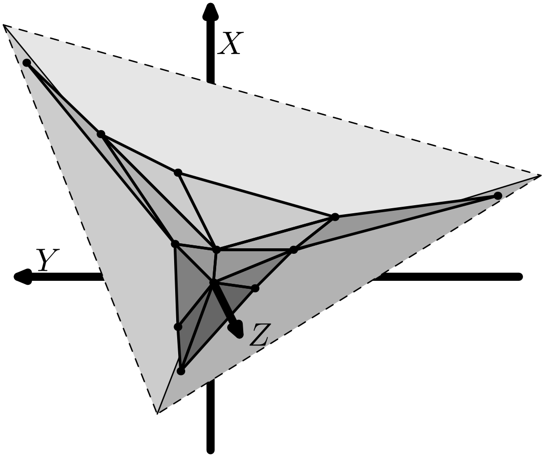
From sail to torus decomposition



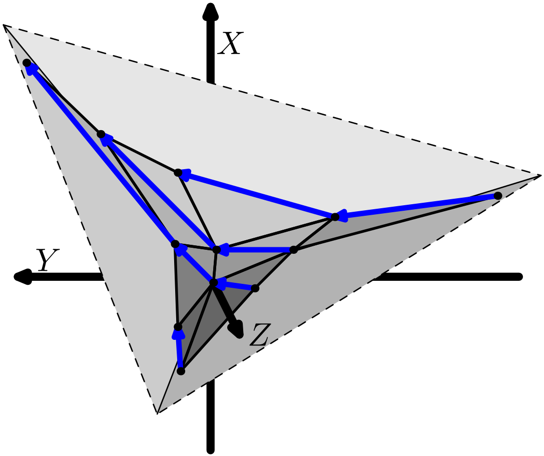
A sail for an algebraic operator *A*.

From sail to torus decomposition



Let $\Xi(A)$ is generated by X and Y.

From sail to torus decomposition



X acts on the sail as a shift.

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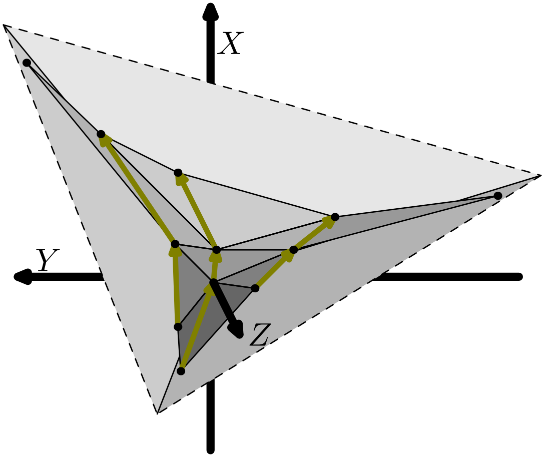

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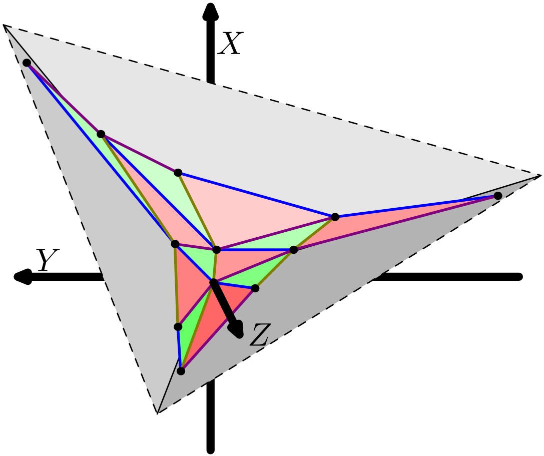
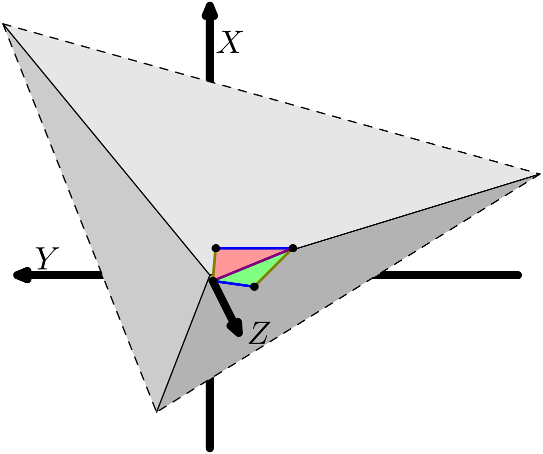


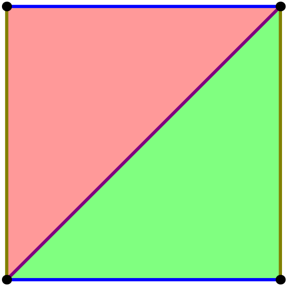
The orbits under the action of $\Xi(A)$.

From sail to torus decomposition



The fundamental domain of the action of $\Xi(A)$.

From sail to torus decomposition



So the factor of the sail under the action of $\Xi(A)$ is a torus.

Operators of multidimensional golden ratio

The following operator defines $(n-1)$-**dimensional golden ratio**:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & \cdots & n-2 & n-1 & n \\ 1 & 2 & \cdots & n-2 & n-1 & n \end{pmatrix}.$$

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For $n=2$ we get the continued fraction defined by the lines

$$y=\frac{1+\sqrt{5}}{2} \quad \text{and} \quad y=\frac{1-\sqrt{5}}{2}.$$

The period is the simplest possible: (1).

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Notice that this operator is **NOT** always irreducible over $\mathbb{Q}$.

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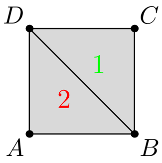
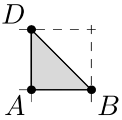
For $n \leq 20$ it is not irreducible for $n =$ 4, 7, 10, 12, 13, 16 ,17, 19.

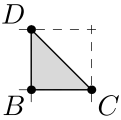
Example 1: 2D golden ratio (E. I. Korkina, G. Lachaud)



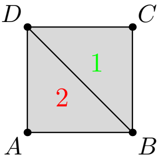
Dirichlet group generators:

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$



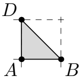


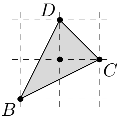
Example 2: (A. D. Bryuno and V. I. Parusnikov)



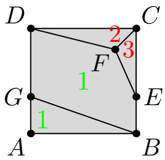
$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Dirichlet group $\Xi(M)$ generators: $X = M^2, \qquad Y = 2I - M^2.$



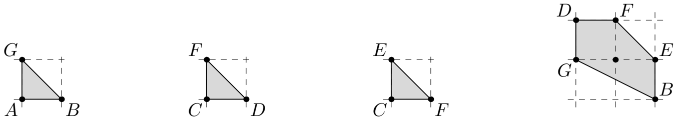


Example 3: (V. I. Parusnikov)



$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -3 \end{pmatrix}.$$

Dirichlet group $\Xi(M)$ generators: $X = M^2, \qquad Y = 3I - 2M^{-1}.$



Decompositions for matrices with small norm

The **norm** is the sum of absolute values of the coefficients.

Theorem

Norm < 5*:*

0 c.f.

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48 generalized golden ratios.

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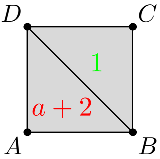
48 generalized golden ratios.

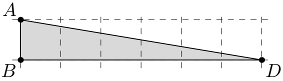
Norm is 6*:*
480 generalized golden ratios;
192 of Example 2;
240 of Example 3.

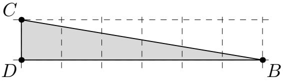
Example 4: (O. Karpenkov)



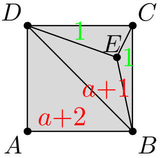
$$M_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1+a-b & -(a+2)(b+1) \end{pmatrix}, \quad a, b \geq 0.$$

Dirichlet group $\Xi(M_{a,b})$ generators: $X_{a,b} = M_{a,b}^{-2}$, $Y_{a,b} = M_{a,b}^{-1}\big(M_{a,b}^{-1} - (b+1)I\big)$.





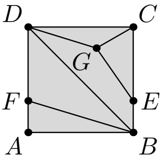
Example 5: (O. Karpenkov)



$$M_a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & a & -2a-3 \end{pmatrix}, \quad a \geq 1.$$

Dirichlet group $\Xi(M_a)$ generators:
$X_a = M_a^{-2}$, $Y_a = \left(2I - M_a^{-2}\right)^{-1}$.

Example 6: (O. Karpenkov)

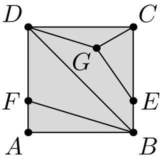


$$M_{a,b} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & (a+2)(b+2)-3 & 3-(a+2)(b+3) \end{pmatrix}, \quad a, b \geq 0.$$

Dirichlet group $\Xi(M_{a,b})$ generators:

$$X_{a,b} = \left((b+3)I - (b+2)M_{a,b}^{-1}\right)M_{a,b}^{-2}, \qquad Y_{a,b} = M_{a,b}^{-2}.$$

Example 6: (O. Karpenkov)



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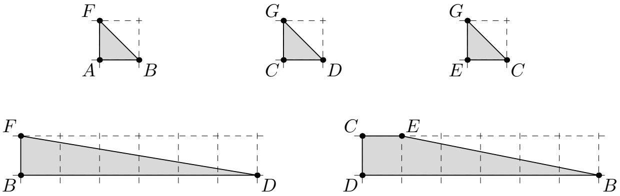
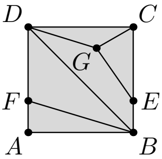


Example 6: (O. Karpenkov)



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The integer distance from the faces to the origin:
from $DBEC$ is 1;
from ABF is 1;
from BFD is 1;
from CDG is $2+2a+2b+ab$;
from CEG is $3+2a+2b+ab$.

Family of Frobenius operators

The family of Frobenius operators

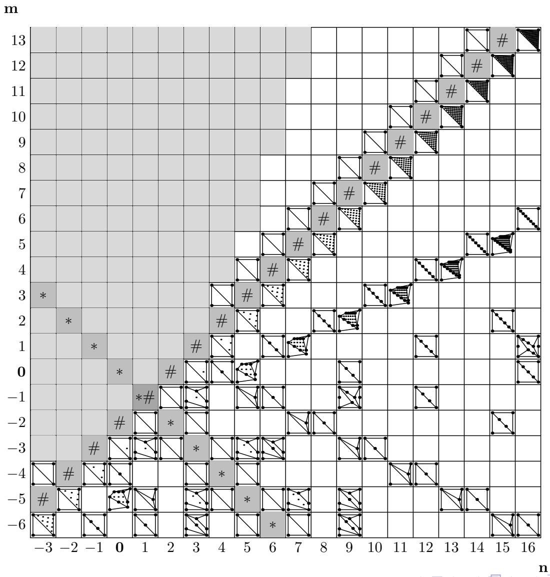
$$A_{m,n} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -m & -n \end{pmatrix},$$

where m and n are integers.

Proposition

The continued fractions for $A_{m,n}$ and $A_{-n,-m}$ are congruent.

Family of Frobenius operators



Questions and problems

Conjecture

(V. I. Arnold) Torus decompositions (affine types of faces and distances to the origin) of noncongruent sails are distinct.

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Any torus decompositions contains a face with unit integer distance to the origin.

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