Adiabatic limits and problems on distribution of integer points

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August, 2010

Adiabatic limits

- (M, g_M) a smooth compact Riemannian manifold.
- $\mathcal{F}$ a smooth foliation on M.
- $F = T\mathcal{F}$ the tangent bundle of $\mathcal{F}$, $H = F^\perp$:

$$TM = F \bigoplus H.$$

- The corresponding decomposition of the metric:

$$g_M = g_F + g_H.$$

- g_ε the one-parameter family of Riemannian metrics on M

$$g_\varepsilon = g_F + \varepsilon^{-2} g_H, \quad \varepsilon > 0.$$

Adiabatic limits

Definition

The Laplace-Beltrami operator defined by $g_\varepsilon, \varepsilon > 0$:

$$\Delta_\varepsilon = d^*_{g_\varepsilon} d,$$

where

- $d : C^\infty(M) \to \Omega^1(M)$ is the de Rham differential;
- $d^*_{g_\varepsilon} : \Omega^1(M) \to C^\infty(M)$ the adjoint of d with respect to the inner products defined by g_ε.

Problem

The limit $\varepsilon \to 0$ — the adiabatic limit.

The problem

For $\varepsilon > 0$, Δ_ε a second order self-adjoint elliptic operator on the compact manifold $M \Longrightarrow$ it has a complete orthonormal system of eigenfunctions

$$\Delta_\varepsilon \varphi_j(\varepsilon) = \lambda_j(\varepsilon)\varphi_j(\varepsilon), \quad \varphi_j(\varepsilon) \in C^\infty(M),$$

where $\lambda_0(\varepsilon) = 0 < \lambda_1(\varepsilon) \leq \lambda_2(\varepsilon) \leq \ldots, \lambda_j(\varepsilon) \to \infty$ as $j \to \infty$.

Problem

The asymptotic behavior of the eigenvalue distribution function

$$N_\varepsilon(\lambda) = \#\{j : \lambda_j(\varepsilon) < \lambda\}, \quad \varepsilon \to 0.$$

or, more generally, of

$$\operatorname{tr} f(\Delta_\varepsilon) = \sum_j f(\lambda_j(\varepsilon)), \quad \varepsilon \to 0.$$

Linear foliations on the two-torus

- $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the two-dimensional torus with the coordinates $(x, y) \in \mathbb{R}^2$, considered modulo integer translations;
- g the Euclidean metric on $\mathbb{T}^2$:

$$g = dx^2 + dy^2.$$

- $\mathcal{F}$ the foliation on $\mathbb{T}^2$ determined by the parallel lines

$$\widetilde{L}_{(x_0,y_0)} = \{(x_0 + t, y_0 + t\alpha) \in \mathbb{R}^2 : t \in \mathbb{R}\}, \quad (x_0, y_0) \in \mathbb{R}^2,$$

  with the slope $\alpha \in \mathbb{R}$.

The adiabatic limit

- The Riemannian metric g_ε is given by

$$g_\varepsilon = \frac{1}{1+\alpha^2}(dx+\alpha dy)^2 + \frac{\varepsilon^{-2}}{1+\alpha^2}(-\alpha dx + dy)^2 .$$

- The corresponding Laplace operator:

$$\Delta_\varepsilon = -\frac{1}{1+\alpha^2}\left(\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}\right)^2 - \frac{\varepsilon^2}{1+\alpha^2}\left(-\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 .$$

The eigenvalue distribution function

Eigenvalues and eigenfunctions

The operator Δ_ε has a complete orthogonal system of eigenfunctions

$$u_{kl}(x,y)=e^{2\pi i(kx+ly)},\quad (x,y)\in\mathbb{T}^2,$$

with the corresponding eigenvalues

$$\lambda_{kl}(\varepsilon)=(2\pi)^2\left(\frac{1}{1+\alpha^2}(k+\alpha l)^2+\frac{\varepsilon^2}{1+\alpha^2}(-\alpha k+l)^2\right),\quad (k,l)\in\mathbb{Z}^2.$$

The eigenvalue distribution function of Δ_ε

$$N_\varepsilon(\lambda)=\text{the number of integer points in the ellipse}$$

$$\{(\xi,\eta)\in\mathbb{R}^2 : (2\pi)^2\left(\frac{1}{1+\alpha^2}(\xi+\alpha\eta)^2+\frac{\varepsilon^2}{1+\alpha^2}(-\alpha\xi+\eta)^2\right)<\lambda\}.$$

The asymptotic formula

Theorem (A. Yakovlev, 2007)

1. For $\alpha \notin \mathbb{Q}$ and $\lambda \in \mathbb{R}$

$$N_\varepsilon(\lambda) = \frac{1}{4\pi}\varepsilon^{-1}\lambda + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

2. For $\alpha = \frac{p}{q} \in \mathbb{Q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ are coprime, and $\lambda \in \mathbb{R}$

$$N_\varepsilon(\lambda) = \varepsilon^{-1} \sum_{k:\frac{4\pi^2}{p^2+q^2}k^2<\lambda} \frac{1}{\pi\sqrt{p^2+q^2}}\left(\lambda - \frac{4\pi^2}{p^2+q^2}k^2\right)^{1/2} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

The semiclassical Weyl formula

- M a compact manifold;
- $V \in C^\infty(M, \mathbb{R})$ a real-valued smooth function;
- the Schrödinger operator

$$H_h = h^2\Delta + V(x), \quad x \in M.$$

- the semiclassical principal symbol of H_h:

$$p(x,\xi) = |\xi|^2 + V(x), \quad (x,\xi) \in T^*M.$$

The semiclassical Weyl formula:

$$\operatorname{tr} f(H_h) = \frac{1}{(2\pi h)^n} \int_{T^*M} f(p(x,\xi))\, dx\, d\xi + o(h^{-n}), \quad h \to 0+.$$

The principal symbol

The operator

$$\Delta_\varepsilon = -\frac{1}{1+\alpha^2}\left(\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial y}\right)^2 - \frac{\varepsilon^2}{1+\alpha^2}\left(-\alpha\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^2.$$

- The conormal bundle

$$N^*\mathcal{F} = \{(x,y,p_2)\in \mathbb{T}^2\times\mathbb{R}\}.$$

- The lifted foliation $\mathcal{F}_N$ on $N^*\mathcal{F}$ is defined by the orbits of the induced flow on $N^*\mathcal{F}$

$$T_\tau(x,y,p_2) = (x+\tau, y+\alpha\tau, p_2), \quad (x,y,p_2)\in\mathbb{T}^2\times\mathbb{R}.$$

- The principal symbol of Δ_ε is a tangentially elliptic operator in $C^\infty(N^*\mathcal{F})$ given by

$$\sigma(\Delta_\varepsilon) = -\frac{1}{1+\alpha^2}\left(\frac{\partial}{\partial x}+\alpha\frac{\partial}{\partial y}\right)^2 + p_2^2.$$

Operator algebras

- Now we assume that $\alpha \notin \mathbb{Q}$.
- The restriction of the operator $\sigma(\Delta_\varepsilon)$ to

$$\tilde{L}_{(x,y,p_2)} = \{(x+\tau, y+\alpha\tau, p_2) \in N^*\mathcal{F} : \tau \in \mathbb{R}\} \cong \mathbb{R}$$

is the second order elliptic differential operator in the space $L^2(\mathbb{R}, \sqrt{1+\alpha^2}d\tau)$:

$$\sigma(\Delta_\varepsilon)_{(x,y,p_2)} = -\frac{1}{1+\alpha^2}\frac{\partial^2}{\partial\tau^2} + p_2^2.$$

- The family $\sigma(\Delta_\varepsilon) = \{\sigma(\Delta_\varepsilon)_{(x,y,p_2)} : (x,y,p_2) \in N^*\mathcal{F}\}$ is affiliated with a certain C^*-algebra $C^*(N^*\mathcal{F}, \mathcal{F}_N)$, called the foliation C^*-algebra of $(N^*\mathcal{F}, \mathcal{F}_N)$.
- The family $e^{-t\sigma(\Delta_\varepsilon)} = \{e^{-t\sigma(\Delta_\varepsilon)_{(x,y,p_2)}} : (x,y,p_2) \in N^*\mathcal{F}\}$ belongs to $C^*(N^*\mathcal{F}, \mathcal{F}_N)$,

Integration over $N^*\mathcal{F}/\mathcal{F}_N$

- The foliation $(N^*\mathcal{F}, \mathcal{F}_N)$ has a natural transverse symplectic structure
- $\Longrightarrow$ it has a natural holonomy invariant transverse measure (a transverse Liouville measure).
- By noncommutative integration theory, there exists the corresponding trace $\operatorname{tr}_{\mathcal{F}_N}$ on $C^*(N^*\mathcal{F}, \mathcal{F}_N)$.
- One can show that

$$\operatorname{tr}_{\mathcal{F}_N} e^{-t\sigma(\Delta_\varepsilon)} < \infty.$$

Integration (continued)

- The kernel of $e^{-t\sigma(\Delta_\varepsilon)_{(x,y,p_2)}}$ in $L^2(\mathbb{R}, \sqrt{1+\alpha^2}d\tau)$

$$K_t(\tau_1,\tau_2) = (4\pi t)^{-1/2} e^{-p_2^2 t} \exp\left(-\frac{(\tau_1-\tau_2)^2}{4t(1+\alpha^2)}\right).$$

- Putting $\tau_1 = \tau_2 = 0$, we get a well-defined function k_t on $N^*\mathcal{F}$:

$$k_t(x,y,p_2) = (4\pi t)^{-1/2} e^{-p_2^2 t}, \quad (x,y,p_2) \in \mathbb{T}^2 \times \mathbb{R}.$$

- Finally, we obtain

$$\operatorname{tr}_{\mathcal{F}_N} e^{-t\sigma(\Delta_\varepsilon)} = \int_{\mathbb{T}^2\times\mathbb{R}} k_t(x,y,p_2)\, dx\, dy\, dp_2 = \frac{1}{2t}.$$

Noncommutative Weyl formula

Fact

$$\operatorname{tr} e^{-t\Delta_\varepsilon} = \int_0^{+\infty} e^{-t\lambda} dN_\varepsilon(\lambda) = \frac{1}{4\pi t}\varepsilon^{-1} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

Theorem

$$\operatorname{tr} e^{-t\Delta_\varepsilon} = \frac{1}{2\pi\varepsilon} \operatorname{tr}_{\mathcal{F}_N} e^{-t\sigma(\Delta_\varepsilon)} + o(\varepsilon^{-1}), \quad \varepsilon \to 0.$$

Some remarks

Remark

For $\alpha \in \mathbb{Q}$: the orbit through (x, y, p_2):

$$\tilde{L}_{(x,y,p_2)} = \{(x+\tau, y+\alpha\tau, p_2) \in N^*\mathcal{F} : \tau \in \mathbb{R}\} \cong S^1.$$

Therefore, the restriction of $\sigma(\Delta_\varepsilon)$ to $\tilde{L}_{(x,y,p_2)}$ is the second order elliptic differential operator $\sigma(\Delta_\varepsilon)_{(x,y,p_2)}$ in the space $L^2(S^1, \sqrt{1+\alpha^2}d\tau)$. So we get a sum over $\mathbb{Z}$.

Remark

Similar formula holds for any Riemannian foliation (Yu. K., 1999).

The setting

- F a p-dimensional linear subspace of $\mathbb{R}^n$;
- $H = F^\perp$ the q-dimensional orthogonal complement of F with respect to the standard inner product $(\cdot,\cdot)$ in $\mathbb{R}^n$, $p+q=n$;
- For any $\varepsilon > 0$, consider the linear transformation $T_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$:

$$T_\varepsilon(x) = \begin{cases} x, & \text{if } x \in F, \\ \varepsilon^{-1}x, & \text{if } x \in H. \end{cases}$$

- For any bounded domain S in $\mathbb{R}^n$ with smooth boundary, we put

$$n_\varepsilon(S) = \#(T_\varepsilon(S) \cap \mathbb{Z}^n), \quad \varepsilon > 0.$$

Problem

The asymptotic behavior of $n_\varepsilon(S)$ as $\varepsilon \to 0$.

Some auxiliary notions

- $\Gamma = \mathbb{Z}^n \cap F$ a free abelian group ($r = \operatorname{rank} \Gamma \leq p$ the rank of Γ)
- For $r \geq 1$, denote by $(\ell_1, \ell_2, \ldots, \ell_r)$ a base in Γ.
- V the r-dimensional subspace of $\mathbb{R}^n$ spanned by $(\ell_1, \ell_2, ... \ell_r)$ (Γ is a lattice in V!).
- Γ^* denote the lattice in V, dual to the lattice Γ:

$$\Gamma^* = \{\gamma^* \in V : (\gamma^*, \Gamma) \subset \mathbb{Z}\}.$$

- For any $x \in V$, we denote by P_x the $(n-r)$-dimensional affine subspace of $\mathbb{R}^n$, passing through x orthogonal to V.

Fact

Γ^* *coincides with the orthogonal projection of* $\mathbb{Z}^n$ *to* V*:*

$$\mathbb{Z}^n \subset \bigcup_{\gamma^* \in \Gamma^*} (P_{\gamma^*} \cap S).$$

Some auxiliary notions

- $Q \subset V$ the parallelepiped spanned by the base $(\ell_1, \ell_2, ...\ell_r)$ in Γ.
- $|Q|$ the r-dimensional Euclidean volume of Q:

$$|Q| = \mathrm{vol}_r(\ell_1, \ell_2, ...\ell_r) = \mathrm{vol}(V/\Gamma).$$

- Remark: for $r = 0$, the groups Γ and Γ^* are trivial, it is natural to put $|Q| = 1$.

The first result

Theorem (Yu.K., A. Yakovlev, 2010)

For any bounded open set S in $\mathbb{R}^n$ with smooth boundary, the formula holds:

$$n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \mathrm{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{\frac{1}{p-r+1}-q}), \quad \varepsilon \to 0. \qquad (1)$$

Remark

In the case when F is trivial, we have $p = r = 0$, $q = n$. The problem is reduced to the classical problem on the asymptotics of the number of integer points in a family of homothetic domains in $\mathbb{R}^n$. Our formula is reduced to the classical formula, going back to Gauss:

$$\#(\varepsilon^{-1} S \cap \mathbb{Z}^n) = \varepsilon^{-n}\mathrm{vol}_n(S) + O(\varepsilon^{1-n}), \quad \varepsilon \to 0.$$

The second result

Theorem (Yu.K., A. Yakovlev, 2010)

For any bounded open set S in $\mathbb{R}^n$ with smooth boundary such that, for any $x \in F$, the intersection $S \cap \{x+H\}$ is strictly convex, the formula holds:

$$n_\varepsilon(S) = \frac{\varepsilon^{-q}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \mathrm{vol}_{n-r}(P_{\gamma^*} \cap S) + O(\varepsilon^{k-q}), \quad \varepsilon \to 0, \qquad (2)$$

where

$$k = \begin{cases} \frac{q+1}{2(p-r+1)} & \text{if } \frac{q-1}{2} \leq p-r \\ \frac{2q}{q+1+2(p-r)} & \text{if } \frac{q-1}{2} > p-r. \end{cases}$$

Remark

In the case when F is trivial, we get $k = 2n/(n+1)$, and the formula is reduced to the following formula:

$$\#(\varepsilon^{-1} S \cap \mathbb{Z}^n) = \varepsilon^{-n}\mathrm{vol}_n(S) + O(\varepsilon^{-n+2-\frac{2}{n+1}}), \quad \varepsilon \to 0.$$

This formula was proved by Randol in 1966.

Example: $n = 2$ and $p = 1$

Fact

Assume that F is the one-dimensional linear subspace of $\mathbb{R}^2$ spanned by $(1, \alpha) \in \mathbb{R}^2$.
For any bounded domain S in $\mathbb{R}^2$ with smooth boundary, we get
1. For $\alpha \notin \mathbb{Q}$,

$$n_\varepsilon(S) = \varepsilon^{-1} \mathrm{area}(S) + O(\varepsilon^{-1/2}), \quad \varepsilon \to 0.$$

2. For $\alpha = \frac{p}{q}$, where p and q are coprime,

$$n_\varepsilon(S) = \varepsilon^{-1} \frac{1}{\sqrt{p^2+q^2}} \sum_{k \in \mathbb{Z}} |S \cap L_k| + O(1), \quad \varepsilon \to 0.$$

The setting

- $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional torus.
- $\mathcal{F}$ a linear foliation on $\mathbb{T}^n$: the leaf L_x of $\mathcal{F}$ through $x \in \mathbb{T}^n$:

$$L_x = x + F \mod \mathbb{Z}^n.$$

Fact

For any $\lambda \in \mathbb{R}$, we have

$$n_\varepsilon(B_{\sqrt{\lambda}}(0)) = N_\varepsilon(4\pi^2\lambda),$$

where $B_{\sqrt{\lambda}}(0)$ is the ball in $\mathbb{R}^n$ of radius $\sqrt{\lambda}$ centered at the origin.

The main result

Theorem (Yu.K., A. Yakovlev, 2010)

For $\lambda > 0$, the following asymptotic formula holds as $\varepsilon \to 0$:

$$N_\varepsilon(\lambda) = \varepsilon^{-q} \frac{\omega_{n-r}}{|Q|} \sum_{\gamma^* \in \Gamma^*} \left(\frac{\lambda}{4\pi^2} - |\gamma^*|^2 \right)^{(n-r)/2} + O(\varepsilon^{k-q}),$$

where ω_{n-r} is the volume of the unit ball in $\mathbb{R}^{n-r}$ and

$$k = \begin{cases} \frac{q+1}{2(p-r+1)}, & \text{if } \frac{q-1}{2} \leq p-r \\ \frac{2q}{q+1+2(p-r)}, & \text{if } \frac{q-1}{2} > p-r. \end{cases}$$