About optimality of Delaunay triangulations

Oleg R. Musin

University of Texas at Brownsville &

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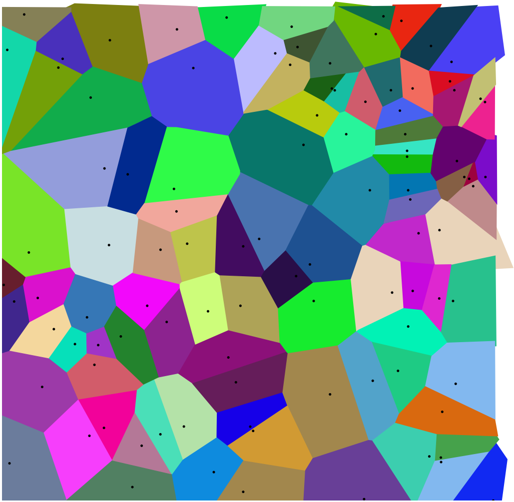
Definitions

Let S be a set of n points $\{p_1, ..., p_n\}$ in $\mathbb{R}^d$. The *Voronoi diagram* is the partition of the $\mathbb{R}^d$ into n convex cells, the Voronoi cells V_i, where each V_i contains all points of the $\mathbb{R}^d$ closer to p_i than to any other point:

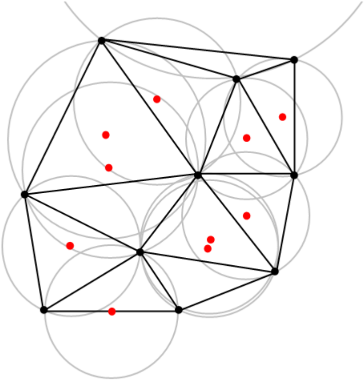
$$V_i = \{x \in \mathbb{R}^d : \forall j \neq i, \mathrm{dist}(x, p_i) \leq \mathrm{dist}(x, p_j)\},$$

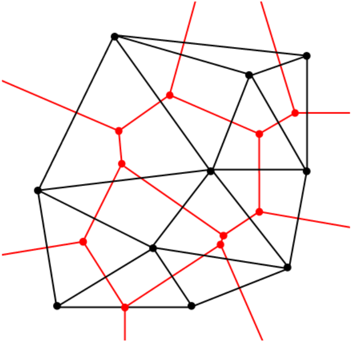
where $\mathrm{dist}(x, y)$ is the Euclidean distance between x and y.
For generic set of n points S in $\mathbf{R}^d$ the straight-line dual of the Voronoi diagram is triangulation of S is called the *Delaunay triangulation* and denoted by $DT(S)$. The $DT(S)$ is triangulation of the convex hull of S in $\mathbb{R}^d$ and set of vertices of $DT(S)$ is S.

Voronoi diagram

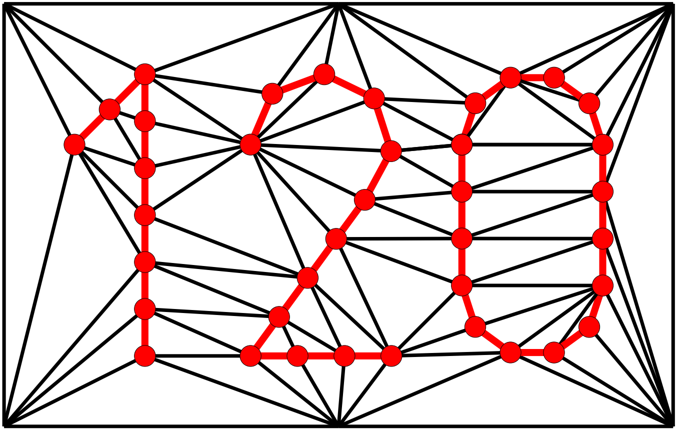


Definitions of the Delaunay triangulation





Delone 120



max-min angle criterion [Sibson, 1978]

Delaunay triangulation is used in numerous applications. For the plane ($d = 2$) it is usually chosen over other triangulations. Often, it is used in 3D case. A logical question may arise: why this triangulation is better than others. Usually, the advantages of Delaunay triangulation are rationalized by the max-min angle criterion. This criterion requires that the diagonal of every convex quadrilateral occurring in the triangulation "should be well chosen", in the sense that replacement of the chosen diagonal by the alternative one must not increase the minimum of the six angles in the two triangles making up the quadrilateral. Thus the Delaunay triangulation of a planar point set maximizes the minimum angle in any triangle. More specifically, the sequence of triangle angles, sorted from sharpest to least sharp, is lexicographically maximized over all such sequences constructed from triangulation of S.

The radii criterion [M., 1993, 1997]

The "radius" functional is the mean of circumradii of triangles for planar triangulations. Let t be a triangulation of S in the plane. Assume that each triangle Δ_i of this triangulation is related to the radius R_i of its circumcircle. Thus every triangulation t is related to the set $\{R_1, ..., R_k\}$ of circumradii of triangles $\Delta_i \in t$. The numbers of triangles for any two triangulations of S are equal, so it is possible to compare sets of radii for different triangulations. In particular, it is possible to compare sums of radii: $\sum R_i$ or power sums: $\sum R_i^a, a > 0$. It seems that triangulation having minimal sum of radii is "better", because all its triangles in "average" are nearer to the regular triangles. *The functional*

$$R(t, a) = \sum_i R_i^a, a > 0$$

attains its minimum iff t is the Delaunay triangulation.

The inradii criterion [Lambert, 1994]

The Delaunay triangulation maximizes the mean inradius: *The functional*

$$r(t) = \sum r_i$$

attains its maximum if and only if t is the Delaunay triangulation.

The harmonic index [M., 1995]

The harmonic index of a triangulation has its origin in the theory of the so called harmonic maps. For polygon P its *harmonic index*

$$\mathrm{hrm}(P) = \sum a_i^2 / S(P),$$

where $a_1, \ldots, a_n$ are the lengths of sides of P and $S(P)$ is its area. For a planar triangulation t of S let denote by $\mathrm{hrm}(t)$ the sum of *hrm* of its triangles:

$$\mathrm{hrm}(t) = \sum_{\Delta_i \in t} \mathrm{hrm}(\Delta_i)$$

The harmonic index hrm(t) *of triangulation* t *of* S *achieves its minimum if and only if* t *is the Delaunay triangulation of* S.

The integral of the gradient squared [Rippa,1990]

The Delaunay triangulation minimizing the integral of the gradient squared.

Bobenko and Springborn [2005] derive the harmonic index theorem from the Rippa theorem.

The minimum area surface [M., 1990]

Let t be a triangulation of $S = \{x_1, \ldots, x_m\} \in \mathbb{R}^2$. For given $y_i, \ldots, y_m$ the set $\{x_i, y_i\}$ is uniquely defined a polygonal surface $F(t, \{y_i\})$ in $\mathbb{R}^3$.

Theorem

For any generic S in the plane there is $\varepsilon > 0$ such that if for all i we have $y_i < \varepsilon$, then the area of $F(t, \{y_i\})$ achieves its minimum iff t is the Delaunay triangulation.

weighted sum of squares of the edge lengths [Rajan, 1994]

For a simplex Δ in $\mathbb{R}^d$ denote by

$$\mathbf{E}(\Delta) := \sum_i |e_i|^2 \operatorname{vol}(\Delta).$$

Then $\mathbf{E}(t) = \sum_i \mathbf{E}(\Delta_i)$ *achieves its minimum if and only if* t *is the Delaunay triangulation.*

The parabolic functional, [M., 1993, 1997, 1998]

$$\mathbf{P}(t) = \sum_i (|x_{i0}|^2 + \ldots + |x_{id}|^2)\,\mathrm{vol}(\Delta_i).$$

The parabolic functional achieves its minimum if and only if t is the Delaunay triangulation.

$$(d+2)\,\mathbf{P}(t) = \mathbf{E}(t) + (d+1)(d+2)\int_{CH(S)} ||x||^2 dx$$

The mean radius

Let t be a triangulation of $\mathrm{CH}(S)$ in $\mathbb{R}^d$. Denote by

$$\mathsf{R}(t, a) := \sum_i R_i^a \operatorname{vol}(\Delta_i).$$

Conjecture: $\mathsf{R}\,(t, a)$, *where* $a \geq 1$, *achieves its minimum if and only if t is the Delaunay triangulation.*

Theorem

The conjecture is correct for $d = 2$.

The D functional

Let

$$\mathsf{D}(t) := \sum_i |b_i - c_i|^2 \operatorname{vol}(\Delta_i),$$

where b_i is the barycenter and c_i is the circumcenter of Δ_i.

Conjecture: *$\mathsf{D}(t)$ achieves its minimum if and only if t is the Delaunay triangulation.*

Theorem

The conjecture is correct for $d = 2$.

Theorem

For any $d \geq 2$ the functional $\mathsf{R}(t, 2) - \mathsf{D}(t)$ attains its minimum if and only if t is the Delaunay triangulation.

Proof.

$$\mathsf{R}(t, 2) - \mathsf{D}(t) = \mathsf{E}(t)/(d+1)^2.$$

□

The Voronoi functional

$$\mathbf{P}(t) - \mathbf{Vr}(t) = c(d)\,\mathbf{R}(t,2) + I(S),$$

where

$$I(S) = (d+1) \int\limits_{CH(S)} ||x||^2 dx, \quad c(d) = \frac{1}{d(d+1)}.$$

Optimal functionals for $d = 2$

1. max-min angle criterion [Sibson, 1978];
2. The radii criterion [M., 1993, 1997];
3. The inradii criterion [Lambert, 1994]
4. The integral of the gradient squared [Rippa,1990];
5. The minimum area surface; [M., 1990]
6. The mean radii: $\mathbf{R}(t, a) := \sum_i R_i^a area(\Delta_i),\ a \geq 1$; [new]
7. The D functional [new].

Optimal functionals for all d

The parabolic functional = (up to constants)=
weighted sum of squares of the edge lengths [Rajan, 1994]

Open problems: to prove conjectures for the R ***and*** D ***functionals.***