

Algebra versus analysis in the theory of flexible polyhedra

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- Two basic theorems of the theory of flexible polyhedra were proven by completely different methods: R. Alexander¹ used analysis, namely, the Stokes theorem, to prove that the total mean curvature remains constant during the flex, while I.Kh. Sabitov² used algebra, namely, the theory of resultants, to prove that the oriented volume remains constant during the flex.
- We show that none of these methods can be used to prove the both theorems.
- As a by-product, we prove that the total mean curvature of any polyhedron in the Euclidean 3-space is not an algebraic function of its edge lengths.

¹Trans. Amer. Math. Soc. **288** (1985), 661–678.

²Fundam. Prikl. Mat. **2** (1996), 1235–1246.

- A polyhedron (more precisely, a polyhedral surface) is said to be **flexible** if its spatial shape can be changed continuously due to changes of its dihedral angles only, i. e., if every face remains self-congruent during the flex.
- In other words, a polyhedron P_0 is flexible if it is included in a continuous family $\{P_t\}$, $0 \leq t \leq 1$, of polyhedra P_t such that, for every t , the corresponding faces of P_0 and P_t are congruent while polyhedra P_0 and P_t are not congruent.
- In general, self-intersections are possible both for P_0 and P_t .
- Without loss of generality we assume that the faces of the polyhedra are triangular.

Existence of flexible polyhedra

- Flexible octahedra with self-intersections in the Euclidean 3-space \mathbb{R}^3 were found and studied by R. Bricard³.
- Flexible self-intersection free sphere-homeomorphic polyhedra in \mathbb{R}^3 were constructed by R. Connelly⁴.
- K. Steffen⁵ constructed a flexible self-intersection free sphere-homeomorphic polyhedron in \mathbb{R}^3 with 9 vertices.

³Journ. de Math. (5) **3** (1897), 113–148.

⁴Inst. Hautes Études Sci. Publ. Math. **47** (1977), 333–338.

⁵*M. Berger*, *Geometry*, I. Springer, Berlin, 1987.

- A citation from Marcel Berger⁶: *“Our opinion is that the known examples of bendable polyhedra do not flex very much.”*
- Given two real numbers $0 < r < R$, there exists a flexible sphere-homeomorphic polyhedron P in \mathbb{R}^3 such that, in one position, P lies in a ball of radius r while, in another position, P is not contained in any ball of radius R .
- There is a lattice tiling in \mathbb{R}^3 such that:
 - (i) each tile P is a sphere-homeomorphic flexible polyhedron and
 - (ii) any flex of P generates a movement of the whole tiling such that the result is a lattice tiling again.
- In \mathbb{R}^3 , there are flexible polyhedra of arbitrary genus as well as non-orientable ones.

⁶A Panoramic View of Riemannian Geometry. Springer, Berlin, 2007. Page 148.

Definiton of the total mean curvature

During last 30 years, many non-trivial properties of flexible polyhedra were discovered. We formulate two of them in a form that is convenient for our purposes.

Definition: Let P be a closed oriented polyhedron in \mathbb{R}^3 , let E be the set of its edges, let $|\ell|$ be the length of edge ℓ , and let $\alpha(\ell)$ be the dihedral angle of P at edge ℓ measured from inside of P . The sum

$$M(P) = \frac{1}{2} \sum_{\ell \in E} |\ell| (\pi - \alpha(\ell))$$

is called the **total mean curvature** of P .

Alexander's theorems

Theorem 1. *Let P_0 be a flexible oriented polyhedron in \mathbb{R}^3 and let $\{P_t\}$, $0 \leq t \leq 1$, be its flex. The total mean curvature $M(P_t)$ is independent of t .*

Remark: Theorem 1 was obtained by R. Alexander⁷ as an obvious corollary of Theorem 2, while the latter was proved with the help of the Stokes theorem, i. e., by means of Analysis.

Theorem 2. *Let P be a closed oriented polyhedron in \mathbb{R}^3 , let \mathbf{w} be its infinitesimal flex, and let $P(t) = \{\mathbf{r} + t\mathbf{w} \mid \mathbf{r} \in P\}$. Then*

$$\left. \frac{d}{dt} \right|_{t=0} M(P(t)) = 0.$$

⁷Trans. Amer. Math. Soc. **288** (1985), 661–678.

Theorem 3. *If $\{P_t\}$ is a flex of an oriented polyhedron in \mathbb{R}^3 , then the oriented volume of P_t is constant in t .*

Remark: Theorem 3 was obtained by I.Kh. Sabitov⁸ as an obvious corollary of Theorem 4, while the latter was proved with the help of the theory of resultants, i. e., by means of Algebra.

Theorem 4. *For the set \mathcal{P}_K of all (not necessarily flexible) closed oriented polyhedra in \mathbb{R}^3 with triangular faces and with a prescribed combinatorial structure K there exists a universal polynomial \mathfrak{p}_K of a single variable whose coefficients are universal polynomials in the edge lengths of a polyhedron $P \in \mathcal{P}_K$ and such that the oriented volume of any $P \in \mathcal{P}_K$ is a root of \mathfrak{p}_K .*

⁸Fundam. Prikl. Mat. **2** (1996), 1235–1246 and Discrete Comput. Geom. **20** (1998), 405–425.

The main results

We found⁹ that Theorem 1 cannot be proved by means of Algebra and Theorem 3 cannot be proved by means of Analysis. More precisely, we proved the following two theorems.

Theorem 5. *There is a closed oriented polyhedron P in \mathbb{R}^3 with the following properties:*

- (i) *the flux across P of some infinitesimal flex of P is non-zero;*
- (ii) *P contains no vertex V whose star lies in a plane;*
- (iii) *P contains no vertex V such that some three edges of P incident to V lie in a plane.*

Theorem 6. *The total mean curvature of any closed oriented polyhedron in \mathbb{R}^3 is not an algebraic function of its edge lengths.*

⁹Aequat. Math. **79**, no. 3 (2010), 229–235.

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Thank you for attention!