Combinatorial diameter of parallelohedra.

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In this talk we will discuss boundaries for the maximal combinatorial diameter of d-dimensional parallelohedra, mostly for space-filling zonotopes.

Definition

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A d-dimensional polytope P is called a *parallelohedron* if the space $\mathbb{R}^d$ can be tiled by parallel copies of P.

Minkowski and Venkov Theorems

Theorem (Minkowski)

A d-dimensional parallelohedron P satisfies the following conditions (Minkowski conditions):

- *P is centrally symmetric;*
- *any facet of P is centrally symmetric;*
- *projection of P along any of its $(d-2)$-face is a parallelogram or centrally symmetric hexagon.*

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Theorem (Venkov)

If polytope P satisfies three Minkowski conditions then P is a parallelohedron.

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In the same way we can define belts for any polytope with centrally symmetric facets. In this case belts can be of arbitrary length.

Distance and Diameter

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The maximal distance between facets of P is called *belt* or *combinatorial* diameter $\mathcal{D}_P$ of parallelohedron P.

The Main Problem

Problem

Find the maximal possible belt diameter $\xi(d)$ of d-dimensional parallelohedron.

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In the case of P is a sum of segments $[0, \mathbf{v}_i]$ we will write $P = Z(V) = Z(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ for $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Examples

Definition

A *permutahedron* Π_d is a convex hull of $(d+1)!$ points $\sigma(1, 2, \ldots, d+1)$ for all permutations σ from the group $\mathcal{S}_{d+1}$. This is d dimensional polytope because all its vertices lies in hyperplane $x_1 + \ldots + x_{d+1} = \dfrac{(d+1)(d+2)}{2}$.

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Permutahedron Π_d is a zonotope and a parallelohedron.

Proposition

Belt diameter of the permutahedron Π_d *is* 2.

Examples II

Example (two-dimensional case)

There are two 2-dimensional parallelohedra, namely parallelogramm and centrally symmetric hexagon. Their combinatorial diameters are equal to 1.

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There are five 3-dimensional paralleohedra, namely cube (or parallelepiped), centrally symmetric hexagonal prism, rhombic dodecahedron, elongated dodecahedron and truncated octahedron.

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Example (three-dimensional case)

There are five 3-dimensional paralleohedra, namely cube (or parallelepiped), centrally symmetric hexagonal prism, rhombic dodecahedron, elongated dodecahedron and truncated octahedron.
The combinatorial diameters of cube and hexagonal prism are equal 1 and belt diameters of all other three-dimensional parallelohedra are equal to 2.

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Conjecture

Belt diameter of every space-filling zonotope is not greater than 2.

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- It is true for small dimensions.

Theorem (A.G., 2010)

The belt diameter of every space-filling zonotope of dimension 4 *or* 5 *is not greater than* 2.

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  And Π_d is "the most complicated" space-filling zonotopes because
  - it has the most possible number of facets for zonotope $(2^{d+1} - 2)$;
  - it has the most possible number of generator vectors for space-filling zonotopes $(\frac{d(d+1)}{2})$.

6-dimensional Example

Theorem (A.G., 2010)

The 6-dimensional zonotope $Z(V)$ *with the vector set*

$$V = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

is a parallelohedron and has belt diameter 3.

Boundary for any d

Theorem (A.G., 2010)

If P is a space-filling zonotope of dimension d then its belt diameter is not greater than $\lceil \log_2 d \rceil$.