The chromatic number of a normed space

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Nelson - Hadwiger problem

- The following problem was posed by Nelson and Hadwiger in 1950:
  what is the minimum number of colors which are needed to paint all the points in $\mathbb{R}^d$ so that any two points at distance 1 apart receive different colors
- Formally,

$$\chi(\mathbb{R}^d) = \min\{m \in \mathbb{N} : \mathbb{R}^d = H_1 \cup \ldots \cup H_m : \forall i, \forall x, y \in H_i |x - y| \neq 1\}.$$

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Some known results

1. 1951, Erdős, de Bruijn: If we accept the axiom of choice, then the chromatic number of the space is equal to the chromatic number of some finite graph, lying in $\mathbb{R}^d$, with edges that connect vertices at unit distance apart
2. $4 \leq \chi(\mathbb{R}^2) \leq 7$
3. 2001, Nechushtan $6 \leq \chi(\mathbb{R}^3) \leq 15$
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Asymptotical lower bound

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- 1972, Larman, Rogers, $\chi(\mathbb{R}^d) \geq c_2 \frac{d^2}{(\log d)^3}$
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- 1981, Frankl, Wilson $\chi(\mathbb{R}^d) \geq (1,207.. + o(1))^d$
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Generalizations and related problems

- In the definition of the chromatic number instead of $\mathbb{R}^n$ with Euclidean metric we can consider an arbitrary space with arbitrary metric. There are a lot of results about the chromatic number of $\mathbb{Q}^d$ and S^d with Euclidean metric, and about the chromatic number of $\mathbb{R}^d_p$.
- We can forbid any set of distances instead of the unit distance. There are works where authors considered maximum of the chromatic number of the space with k forbidden distances among all k-element sets.
- We also can consider colorings of the space where each color is of certain type: for example, each color is measurable or each color is a disjoint union of polyhedra. We have $5 \le \chi(\mathbb{R}^2) \le 7$ if each color is measurable.

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Asymptotical bounds

1. We denote by $\chi(\mathbb{R}^d_K, A)$ the chromatic number of the space with norm induced by convex centrally symmetric bounded body K and with set A of forbidden distances. By $\chi(\mathbb{R}^d_p)$ we denote the chromatic number of the space with l_p-norm and with one forbidden distance.
2. For $p = 2$ (classical case) we have $(1,237.. + o(1))^d \leq \chi(\mathbb{R}^d_2) \leq (3 + o(1))^d$. The upper bound is due to Larman, Rogers, 1972.
3. We have $\chi(\mathbb{R}^d_p) \geq (1,207... + o(1))^d$, $\chi(\mathbb{R}^d_\infty) = 2^d$ and $\chi(\mathbb{R}^d_1) \geq (1,369... + o(1))^d$. Last result is due to Raigorodskii.

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New results

1. Furedi and Kang in several works established different upper estimates on the $\chi(\mathbb{R}^d_p)$. The best one is $\chi(\mathbb{R}^d_K) \le cd(\ln d)5^d$ (2008)
2. **Theorem 1**. *Let $\mathbb{R}^d_K$ be a normed space. Then*

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1. We can improve this theorem even more in the case of the l_p-space $\mathbb{R}_p^d$.
Theorem 2. *Let $\mathbb{R}_p^d$ be a normed space. Then*

$$\chi(\mathbb{R}_p^d) \leq 2^{(1+c_p+\delta_d)d},$$

where $\delta_d \to 0$ when $d \to \infty$, and $c_p < 1$ when $p > 2$ and $c_p \to 0$ when $p \to \infty$.
Indeed, for $p(d) > \omega(d) d \ln \ln d$, where $\omega(d)$ is a function, which tends to infinity arbitrary slow, we can obtain the bound $\chi(\mathbb{R}_{p(d)}^d) \leq (\ln d + \ln \ln d + \ln 2 + 1 + o(1))d2^d = (2+o(1))^d$.
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   Indeed, for $p(d) > \omega(d) d \ln\ln d$, where $\omega(d)$ is a function, which tends to infinity arbitrary slow, we can obtain the bound $\chi(\mathbb{R}^d_{p(d)}) \le (\ln d + \ln\ln d + \ln 2 + 1 + o(1)) d 2^d = (2+o(1))^d$.
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Main ingredients of the proof of theorems 1 and 2

1. At first we prove a slight variation of the Erdős - Rogers theorem (1962) about covering the space by copies of convex bodies
2. We use result of Schmidt (1963), that strengthen famous Minkovskiy - Hlawka theorem.
3. In the theorem 2 we also use a result of Odlyzko, Rush concerning packing of superballs.
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1. In addition, I proved the theorem concerning the chromatic number of the space with the segment of forbidden distances. Let $A = [1, l]$.
2. **Theorem 3.** *Let $\mathbb{R}^d_K$ be a normed space.*
   1. *Then* $\chi(\mathbb{R}^d_K, A) \leq (2(l+1) + o(1))^d$.
   2. *Let* $p > 2$. *Then* $\chi(\mathbb{R}^d_p, A) \leq (2^{c_p}(l+1) + o(1))^d, c_p < 1, c_p \to 0$ *when* $p \to \infty$.
   3. *Let* $l \geq 2$. *Then* $\chi(\mathbb{R}^d_K, A) \geq (l/2)^d$.
   4. *Let* $l \geq 2$. *Then* $\chi(\mathbb{R}^d_p, A) \geq (b \cdot l)^d$, *where* $b = \frac{\sqrt[p']{2}}{2}$ *and* $p' = \max\{p, \frac{p}{p-1}\}$.
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2. From the other side, best known lower bounds on the chromatic number of the space with k forbidden distances are obtained on the set $A_0 = \{\sqrt{2p}, \ldots, \sqrt{2kp}\}$, where p is a certain prime number.
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3. Additional ingredients are famous Kabatyanskiy – Levenstein bound and Pitchugov's bound on the radius of Jung's ball in $\mathbb{R}_p^d$.

Proof of the theorem 3

1. Proof of the upper bounds is quite similar to the proof of the theorems 1 and 2.
2. The technique, used to obtain lower bounds, is also based on a construction of some packing.
3. Additional ingredients are famous Kabatyanskiy – Levenstein bound and Pitchugov's bound on the radius of Jung's ball in $\mathbb{R}_p^d$.

The end

Thank You