On axiomatic parametrization

Advantages:

- Common terminology.
- Spaces classification.
- Comparison of space properties.
- Construction of new geometries .
- Theorem depending on parameters.
- Equations depending on parameters.

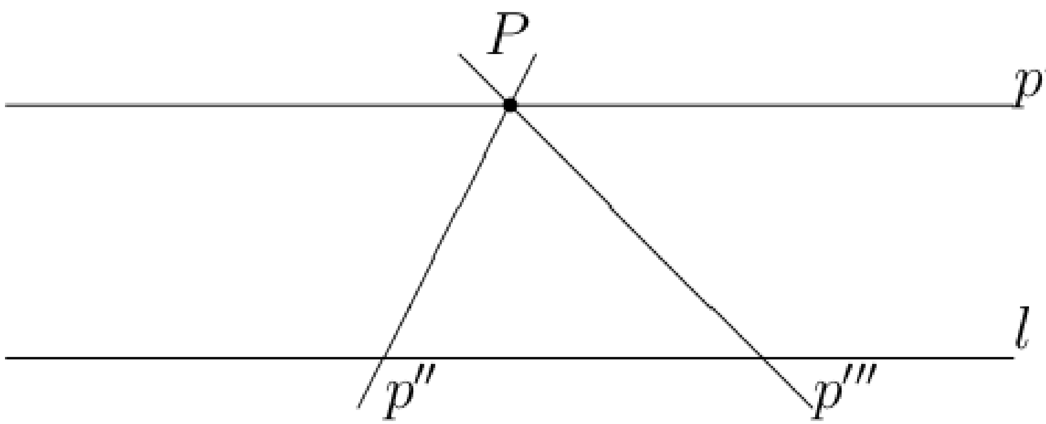
The problem

- Geometry primitives differ from one geometry to another.
- Development of further dimension axiomatic is non–trivial.
- Construction of non–euclidean axiomatic is non–trivial.
- Euclidean space of any dimension is studied using vector space model.
- Good axiomatic for many geometries was developed after a good model was elaborated and geometric properties were well studied.

Homogeneous Spaces

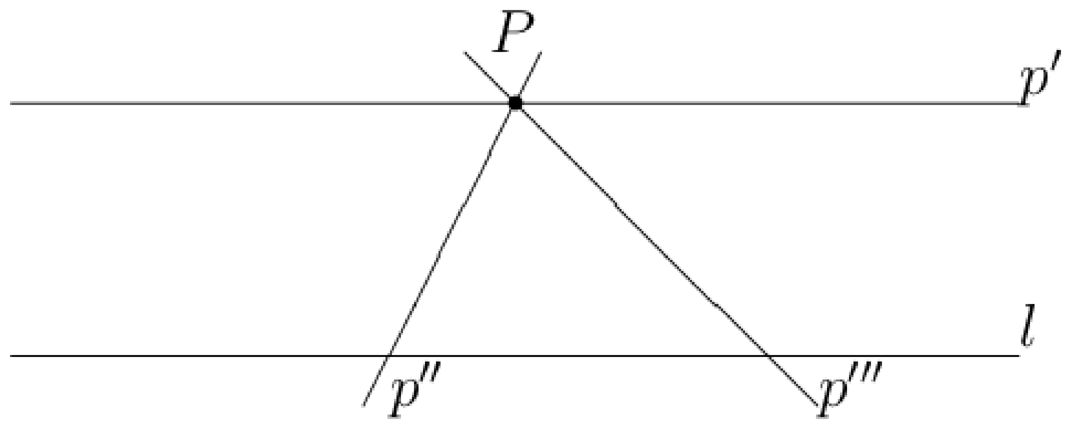
- Homogeneous spaces are spaces that looks the same everywhere. All their elements of same order: points, lines, planes are congruend.
- Euclidean, hyperbolic, elliptic spaces are all homogeneous.
- Much more spaces are also homogeneous.

Parallel Postulate (linear)



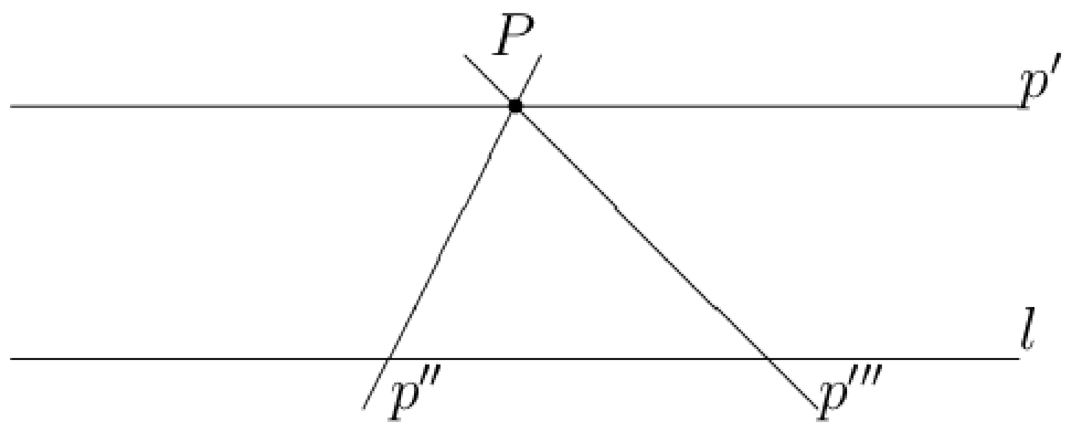
For a given line ***l*** and a point ***P*** not in ***l***, exists one line ***p*** containing ***P*** so that ***p*** doesn't intersect ***l***.

Parallel Postulate (hyperbolic)



For a given line ***l*** and a point ***P*** not in ***l***, exist at least two lines ***p'*** and ***p''*** containing ***P*** so that ***p'*** doesn't intersect ***l***, and ***p''*** doesn't intersect ***l***.

Parallel Postulate (elliptic)



For a given line ***l*** and a point ***P*** not in ***l***, exist no lines ***p*** containing ***P*** so that ***p*** doesn't intersect ***l***.

Or:

For a given line ***l*** and a point ***P*** not in ***l***, all lines ***p*** containing ***P*** intersect ***l***.

Parametrizied Parallel Postulate

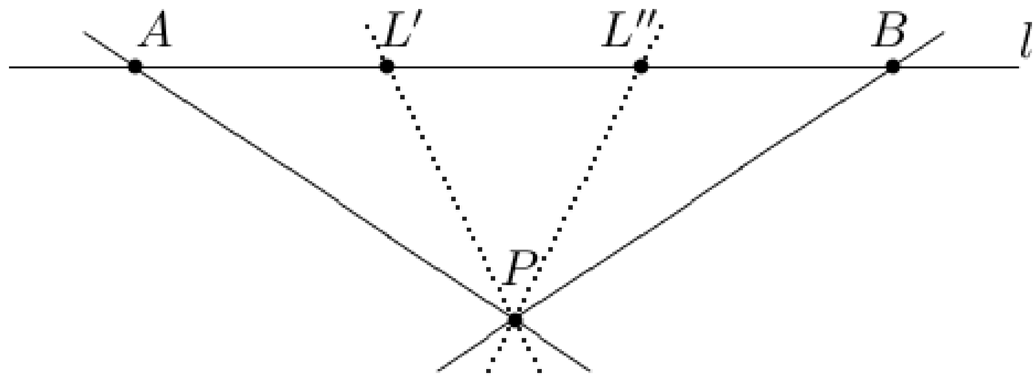
For a given line ***l*** and a point ***P*** not in ***l***, exist 0^k lines ***p*** containing ***P*** so that ***p*** doesn't intersect ***l***.

k = 1	**k = 0**	**k = -1**
Elliptic	Linear	Hyperbolic

Principle of Duality

$$\begin{aligned}
\text{line } l \quad &\longleftrightarrow \quad \text{point } L, \\
P \in l \quad &\longleftrightarrow \quad p \text{ contains } L, \\
AB = \varphi \quad &\longleftrightarrow \quad \angle ab = \varphi, \\
a \cap b = C \quad &\longleftrightarrow \quad c = AB, \\
a \cap b = \emptyset \quad &\longleftrightarrow \quad A \text{ is unconnectable with } B.
\end{aligned}$$

Connectable Postulate (elliptic)

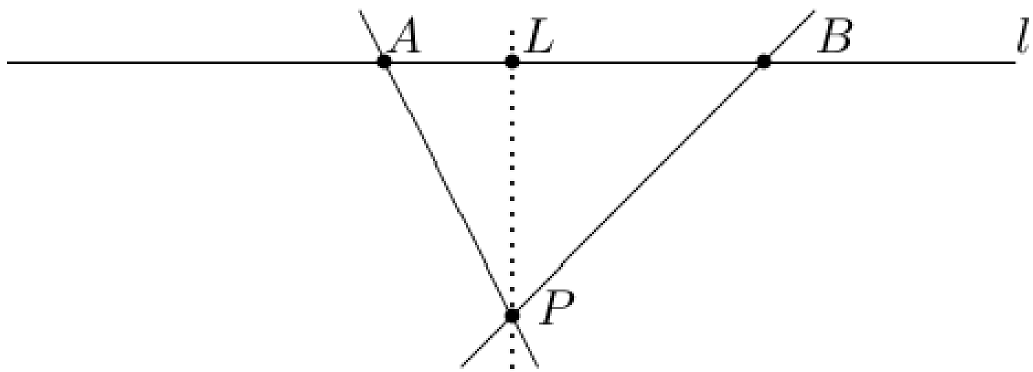


On a line ***l*** not containing ***P*** exist no points ***L*** unconnectable with ***P***.

Or:

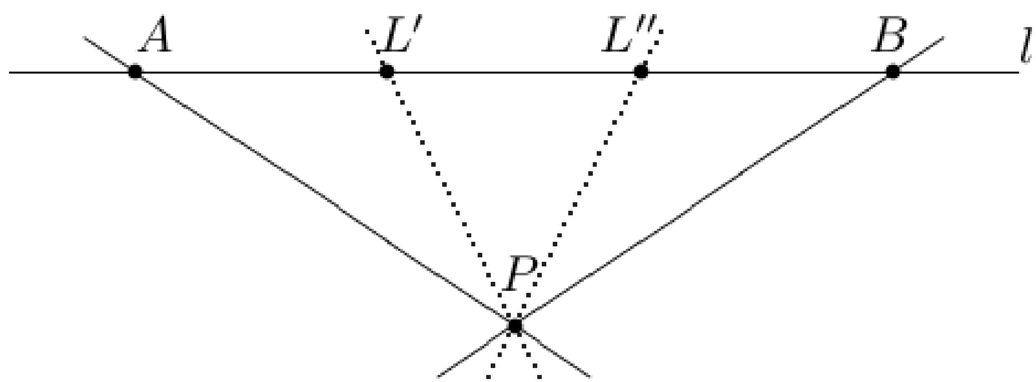
All points ***L*** of line ***l*** are connectable with some point ***P*** not in ***l***.

Connectable Postulate (linear)



On a line ***l*** not containing ***P*** exists one point ***L*** unconnectable with ***P*** (Galilean geometry).

Connectable Postulate (hyperbolic)



On a line ***l*** not containing ***P*** exist at least two points ***L'*** and ***L''*** unconnectable with ***P*** (Minkowskii geometry).

Classification of Plane Homogenious Geometries

On a line *l* not containing ***P*** exist 0^{k2} points ***L*** unconnectable with ***P***.

	k1 = 1	**k1 = 0**	**k1 = -1**
k2 = 1	Elliptic	Linear	Hyperbolic
k2 = 0	*Anti-Euclidean*	Galilean	*Anti-Minkowskii*
k2 = -1	*Anti-hyperbolic*	Minkowskii	*???*

Functions *C(x)*, *S(x)*, *T(x)*

Depending on value of *k*, consider the following functions:

$$C(x)=\sum_{i=0}^{\infty}(-k)^i\frac{x^{2i}}{(2i)!}$$

$$S(x)=\sum_{i=0}^{\infty}(-k)^i\frac{x^{2i+1}}{(2i+1)!}$$

$$T(x)=\frac{S(x)}{C(x)}$$

$$C(x)=\begin{cases}\cos x,\ k=1\\ 1,k=0\\ \cosh x,\ k=-1\end{cases}$$

$$S(x)=\begin{cases}\sin x,\ k=1\\ x,\ k=0\\ \sinh x,\ k=-1\end{cases}$$

$$T(x)=\begin{cases}\tan x,\ k=1\\ x,\ k=0\\ \tanh x,\ k=-1\end{cases}$$

Having several characteristics k_1, ..., k_n, consider respective functions *C*, *S* and *T*.

Main Rotations

Consider n-dimensional projective space $\boldsymbol{RP}^n$ with vectors $x = [x_0:...:x_n]$, $y = [y_0:...:y_n] \in \boldsymbol{RP}^n$ using homogenious coordinates. Name *main rotations* the following mappings (affecting i-1 and i rows):

$$R_i(x)=\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & C_i(x) & -k_i S_i(x) & \cdots & 0 \\ 0 & \cdots & S_i(x) & C_i(x) & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Dot Product

Consider dot product of vectors $x = [x_0 : \ldots : x_n]$, $y = [y_0 : \ldots : y_n] \in \boldsymbol{RP}^n$:

$$x \circ y = \sum_{i=0}^{n} K_i x_i y_i ,$$

$$K_i = \prod_{j=1}^{i} k_j , i = \overline{0,n}$$

It can be verified that all main rotations preserve this product. That is:

$$x \circ y = R_i(a) x \circ R_i(a) y , \forall x , y \in RP^n , a \in R , i \in \overline{1,n}$$

Space definition

- Consider unite sphere $B^n = \{x \in \boldsymbol{RP}^n, x \circ x = 1\}$.
- It's easy to see that all main rotations preserve sphere B^n.
- Define n-dimensional space $\boldsymbol{B}^n$ with specification $\{k_1, ..., k_n\}$ as respective unite sphere B^n.
- It is easy to see that for $m < n$, $\boldsymbol{B}^m \subset \boldsymbol{B}^n$.
- Define points $X \in \boldsymbol{B}^n$ as their corresponding vectors $x \in B^n$.
- Define motion as product of main rotations.
- Define m-dimensional planes of $\boldsymbol{B}^n$ images of $\boldsymbol{B}^m$ on some motion.

Measure Definition

- For origin $O = [1:0:...:0]$ and some point $X = R_1(a)O \in \boldsymbol{B}^1$ define distance $|OX|$ as parameter a of the first main rotation.
- For subspace $\boldsymbol{B}^m \subset \boldsymbol{B}^n$ and some m-dimensional plane $p^m = R_{m+1}(a)\boldsymbol{B}^m \subset \boldsymbol{B}^{m+1}$ define (m+1)-dimensional (dihedral) angle as parameter a of m+1 main rotation.
- It can be demonstrated that all motions preserve all measures.

Right triangle equations

Global	Elliptic	Euclidean	Hyperbolic
$T(b) = T(c) \cos\alpha$	$\text{tg } b = \text{tg } c \cos\alpha$	$b = c \cos\alpha$	$\text{th } b = \text{th } c \cos\alpha$
$T(a) = T(c) \cos\beta$	$\text{tg } a = \text{tg } c \cos\beta$	$a = c \cos\beta$	$\text{th } a = \text{th } c \cos\beta$
$S(a) = S(c) \sin\alpha$	$\sin a = \sin c \sin\alpha$	$a = c \sin\alpha$	$\text{sh } a = \text{sh } c \sin\alpha$
$S(b) = S(c) \sin\beta$	$\sin b = \sin c \sin\beta$	$b = c \sin\beta$	$\text{sh } a = \text{sh } c \sin\beta$
$T(a) = S(b) \text{ tg } \alpha$	$\text{tg } a = \sin b \text{ tg } \alpha$	$a = b \text{ tg } \alpha$	$\text{th } a = \text{sh } b \text{ tg } \alpha$
$T(b) = S(a) \text{ tg } \beta$	$\text{tg } b = \sin a \text{ tg } \beta$	$b = a \text{ tg } \beta$	$\text{th } b = \text{sh } a \text{ tg } \beta$
$\cos\beta = C(b) \sin\alpha$	$\cos\beta = \cos b \sin\alpha$	$\cos\beta = \sin\alpha$	$\cos\beta = \text{ch } b \sin\alpha$
$\cos\alpha = C(a) \sin\beta$	$\cos\alpha = \cos a \sin\beta$	$\cos\alpha = \sin\beta$	$\cos\alpha = \text{ch } a \sin\beta$
$\text{tg } \alpha \text{ tg } \beta \, C(c) = 1$	$\text{tg } \alpha \text{ tg } \beta \cos c = 1$	$\text{tg } \alpha \text{ tg } \beta = 1$	$\text{tg } \alpha \text{ tg } \beta \text{ ch } c = 1$
$C(c) = C(a)\, C(b)$	$\cos c = \cos a \cos b$		$\text{ch } c = \text{ch } a \text{ ch } b$
$T^2(c) = T^2(a) + T^2(b) + k1\, T^2(a)\, T^2(b)$	$\text{tg}^2 c = \text{tg}^2 a + \text{tg}^2 b + \text{tg}^2 a \text{ tg}^2 b$	$c^2 = a^2 + b^2$	$\text{th}^2 c = \text{th}^2 a + \text{th}^2 b - \text{th}^2 a \text{ th}^2 b$

Generic triangle equations

$$\frac{S_1(a)}{S_2(\alpha)} = \frac{S_1(b)}{S_2(\beta)} = \frac{S_1(c)}{S_2(\gamma)}$$

$$C_1(a) = C_1(b)C_1(c) + k_1 S_1(b) S_1(c) C_2(\alpha),$$

$$T_1^2(a) = \frac{T_1^2(b) + T_1^2(c) - 2\mathrm{T}_1(b)T_1(C)C_2(\alpha) + k_1 k_2 T_1^2(b) T_1^2(c) S_2^2(\alpha)}{(1 + k_1 T_1(b) T_1(c) C_2(\alpha))^2}$$

$$C_2(\alpha) = C_2(\beta')C_2(\gamma) + k_2 S_2(\beta') S_2(\gamma) C_1(a),$$

$$T_2^2(\alpha) = \frac{T_2^2(\beta') + T_2^2(\gamma) - 2\mathrm{T}_2(\beta')T_2(\gamma)C_1(a) + k_1 k_2 T_2^2(\beta') T_2(\gamma) S_1^2(a)}{(1 + k_2 T_2(\beta') T_2(\gamma) C_1(a))^2}$$

Example: Minkowskii Space-time

•Minkowskii space-time M^4 is 4-dimensional linear space. A point $P \in M^4$ with coordinates $P = (t, x, y, z)$. Invariant quadric form is $t^2 - x^2 - y^2 - z^2$.
•This linear space have specification {0, -1, 1, 1}. Its sphere B^4 has equation (*i*) using homogeneous coordinates [p:t:x:y:z]. Time *pt* is linear and space *pxyz* is linear.

$$K_0 p^2 + K_1 t^2 + K_2 x^2 + K_3 y^2 + K_4 z^2 = 1$$
$$p^2 + 0t^2 + 0x^2 + 0y^2 + 0z^2 = 1 \, (i)$$

Curved Minkowskii Space-time

- If $k_1 = 1$ then equation changes to (*ii*), having elliptic time (overall period is πr_t) and hyperbolic space.
- If $k_1 = -1$ then equation changes to (*iii*), having hyperbolic time and elliptic space (total volume is $\pi^2 r_s^3$).

$$p^2 + t^2 - x^2 - y^2 - z^2 = 1 \,(ii)$$
$$p^2 - t^2 + x^2 + y^2 + z^2 = 1 \,(iii)$$