On the polyhedral product functor: a method of decomposition for moment-angle complexes

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joint work with (1) Tony Bahri, Martin Bendersky, and Sam Gitler in addition to (2) Clark Haynes, and Dan Koditschek

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The setting and the problems:

- This report addresses joint work with Tony Bahri, Martin Bendersky, and Sam Gitler in addition to a an engineering question in joint work with Dan Koditschek, and Clark Haynes. Much of this report gives a picture of 'how and where' some mathematical structures fit together and covers the following:

- Generalized moment-angle complexes (polyhedral product functors), definitions and basic properties, as well as
- connections to questions in engineering robotics, generalized moment-angle complexes as "spaces of robot legs".

Ingredients:

- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)_{i=1}^m\}$ denote a set of triples of CW–complexes with base-point x_i in A_i.
- Let K denote an abstract simplicial complex with m vertices labeled by the set

$$[m] = \{1, 2, \ldots, m\}.$$

- Thus, a $(k-1)$-simplex σ of K is given by an ordered sequence

$$\sigma = (i_1, \cdots, i_k)$$

  with $1 \leq i_1 < \cdots < i_k \leq m$ such that if $\tau \subset \sigma$, then τ is required to be a simplex of K.
- In particular the empty set $\emptyset$ is a subset of σ and so it is in K. Define the length of I by the formula $|I| = k$.

Definitions:

- As above, let (i) $(\underline{X}, \underline{A})$ denote the collection $\{(X_i, A_i, x_i)_{i=1}^m\}$ and (ii) K denote a simplicial complex. The *generalized moment-angle complex or polyhedral product functor* determined by $(\underline{X}, \underline{A})$ and K denoted

$$Z(K; (\underline{X}, \underline{A}))$$

is defined as follows:

- For every σ in K, let $D(\sigma) = \prod_{i=1}^m Y_i, \quad \text{where}$
  - $Y_i = X_i$ if $i \in \sigma$, and
  - $Y_i = A_i$ if $i \in [m] - \sigma$.
- The generalized moment-angle complex is

$$Z(K; (\underline{X}, \underline{A})) = \bigcup_{\sigma \in K} D(\sigma) = \text{colim}_{\sigma \in K} D(\sigma).$$

Remarks:

- Generalized moment-angle complexes have been studied by topologists since the 1960's thesis of G. Porter. In the 1970's E. B. Vinberg developed some of their features. In the late 1980's S. Lopez de Medrano developed beautiful properties of intersections of quadrics with recent further developments in joint work with S. Gitler.
- In seminal work during the early 1990's, M. Davis and T. Januszkiewicz introduced manifolds now often called quasi-toric manifolds. They showed that every quasi-toric manifold is the quotient of a moment-angle complex by the free action of a real torus. The moment-angle complex is denoted $Z(K;(D^2,S^1))$ where K is a finite simplicial complex.

Remarks continued:

- The integral cohomology of the spaces $Z(K;(D^2, S^1))$ has been studied by Goresky-MacPherson, Buchstaber-Panov, Panov, Baskakov, Hochster, Denham-Suciu, and Franz. Among others who have worked extensively on generalized moment-angle complexes are Notbohm-Ray, Grbic-Theriault, Strickland and Kamiyama-Tsukuda.
- Buchstaber-Panov synthesized several different, important developments in this subject.
- In the special case where $X_i = X$ and $A_i = A$ for all $1 \leq i \leq m$, it is convenient to denote the generalized moment-angle complex by $Z(K;(X, A))$ to coincide with the notation in work of Graham Denham and Alex Suciu, who inspired much of the work here.

Examples:

- **Example 1:**
- Let K denote the 2-point complex $\{1,2\}$ with $(X,A)=(D^1,S^0)$. Then

$$Z(K;(D^1,S^0))=(D^1\times S^0)\cup(S^0\times D^1)=S^1.$$

- Let K denote the 2-point complex $\{1,2\}$ with $(X,A)=(D^2,S^1)$. Then

$$Z(K;(D^2,S^1))=(D^2\times S^1)\cup(S^1\times D^2)=S^3.$$

More examples:

- More generally, $Z(K;(D^2,S^1))$ has the homotopy type of the complement of the union of certain coordinate planes in $\mathbb{C}^m$ corresponding to 'coordinate subspace arrangements' as described next.
- Given a simplicial complex K with m vertices, and a simplex $\omega \in \Delta[m-1]$, define

$$L_\omega = \{(z_1, \cdots, z_m) \in \mathbb{C}^m |\ z_{i_1} = \cdots = z_{i_k} = 0\}$$

for

$$\omega = (i_1, \cdots, i_k).$$

Complements of coordinate planes:

Definition:

$$U(K) = \mathbb{C}^m - \bigcup_{\omega \notin K} L_\omega$$

The following is one elegant result due to Victor Buchstaber and Taras Panov.

Proposition: The natural inclusion

$$Z(K; (D^2, S^1)) \to U(K)$$

is an $(S^1)^m$-equivariant strong deformation retract.

Examples continued:

- In addition, $(S^1)^m = T^m$ acts naturally on the product $(D^2)^m$ and on $Z(K;(D^2,S^1))$. The Davis-Januszkiewicz space is the associated Borel construction

$$\mathcal{DJ}(K) = ET^m \times_{T^m} Z(K;(D^2,S^1)).$$

- A special case of a beautiful theorem of Denham-Suciu gives that $\mathcal{DJ}(K)$ is homeomorphic to

$$Z(K;(\mathbb{CP}^\infty,*)).$$

Examples continued:

- The generalized moment-angle complex $Z(K;(S^1,*))$ are examples of spaces listing positions of robotic 'legs' as illustrated next.
- The following is a moving example of $Z(K;(S^1,*))$ 'upstairs'.

Initial structure theorems (from the eyes of homotopy theory):

- The purpose of the next few sections is to provide structure for the generalized moment-angle complex after suspending the space.
- The motivation is partially homological as well as computational and arises from a geometric decomposition.
- The basic decompositions arise from the suspension of a space X is given by

$$\Sigma(X) = C_+(X) \cup C_-(X)$$

  where $C_+(X)$ is the "upper cone" and $C_-(X)$ is an "lower cone" glued together along X.

Technical point regarding base-points:

- Notice that $\Sigma(X)$ does not have a natural base-point. This is remedied by using the 'reduced suspension'

$$C_+(X) \cup C_-(X)/([0,1] \times *_X).$$

The language of wedge products, and smash products:

- Let $(X, *_X)$ and $(Y, *_Y)$ be pointed CW complexes.
- The wedge product (or 'wedge' in the vernacular)

$$X \vee Y$$

is the subspace of the product $X \times Y$ given by

$$X \vee Y = (X \times *_Y) \cup (*_X \times Y).$$

Wedge products, and smash products continued:

- The smash product (or 'smash' in the vernacular)

$$X \wedge Y$$

is the quotient space of the product $X \times Y$ given by

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

Examples of smash products:

- The smash product
$$S^p \wedge S^q$$
is homeomorphic equivalent to
$$S^{p+q}.$$

- The smash product
$$S^1 \wedge X$$
is homeomorphic to the suspension
$$\Sigma(X).$$

Elementary properties of suspensions, wedge products, and smash products:

- Let X be a connected CW complex, and

$$H_i(X)$$

the i-th homology group of X. Then there are natural isomorphisms:

$$\Delta : H_{i+1}(\Sigma X) \to H_i(X)$$

for all $i > 0$.

Elementary properties of suspensions, wedge products, and smash products continued:

- If X and Y are pointed CW-complexes, there are natural (pointed) homotopy equivalences

$$\Sigma(X \vee Y \vee (X \wedge Y)) \to \Sigma(X \times Y).$$

- Thus there is a homotopy equivalance

$$S^{p+1} \vee S^{q+1} \vee S^{p+q+1} \to \Sigma(S^p \times S^q).$$

- The spaces $X \vee Y \vee (X \wedge Y)$, and $X \times Y$ are usually not homotopy equivalent as they usually have different cup product structures in cohomology.

Smash moment-angle complexes and their applications:

The purpose of the next few slides is to describe the structure of the suspension of moment-angle complexes

$$Z(K;(\underline{X},\underline{A}))$$

in terms of 'smash moment-angle complexes' to be made precise next. This information is then applied to obtain information about

- homology,
- cohomology for various cohomology theories in addition to singular cohomology, and
- the cup product structure for the cohomology ring of the moment-angle complex.

‘Smash moment-angle complexes’:

- Recall that moment-angle complexes are subspaces of product spaces.
- Passing to the ‘world’ of pointed spaces (in which all maps are required to preserve base-points), there are natural analogues called

  ‘smash moment-angle complexes’

  where all products in the earlier definition are replaced by smash products.

‘Smash moment-angle complexes’ continued:

- Given $(\underline{X}, \underline{A}, \underline{*})$, and a simplicial complex K with m vertices, recall that

$$Z(K; (\underline{X}, \underline{A}))$$

is a subspace of the product

$$X_1 \times X_2 \times \cdots \times X_m.$$

- Define the ‘smash moment-angle complex’

$$\widehat{Z}(K; (\underline{X}, \underline{A}))$$

to be the image of $Z(K; (\underline{X}, \underline{A}))$ in the smash product

$$X_1 \wedge X_2 \wedge \cdots \wedge X_m.$$

Language:

- Let K denote a simplicial complex with m vertices. Given a sequence

  $$I = (i_1, \cdots, i_k)$$

  with $1 \leq i_1 < \cdots < i_k \leq m$, define $K_I \subseteq K$ to be the

  full sub-complex

  of K consisting of all simplices of K which have all of their vertices in I, that is $K_I = \{\sigma \cap I | \sigma \in K\}$.

A decomposition:

- **Theorem 1**
  Let K be an abstract simplicial complex with m vertices. Given $(\underline{X}, \underline{A}) = \{(X_i, A_i)_{i=1}^m\}$ where (X_i, A_i, x_i) are pointed triples of CW-complexes, there is a natural pointed homotopy equivalence

$$H : \Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma(\bigvee_{I \subseteq [m]} \widehat{Z}(K_I; (\underline{X_I}, \underline{A_I}))).$$

Partial orderings:

- Let K denote a simplicial complex.
- There is a partially ordered set (**poset**) $\bar{K}$ associated to any simplicial complex K as follows. A point σ in $\bar{K}$ corresponds to a simplex $\sigma \in K$ with order given by *reverse* inclusion of simplices.
- Thus $\sigma_1 \leq \sigma_2$ in $\bar{K}$ if and only if $\sigma_2 \subseteq \sigma_1$ in K.
- The empty simplex $\emptyset$ is the unique maximal element of $\bar{K}$. Let P be a poset with $p \in P$.
- There are further posets given by

$$P_{\leq p} = \{q \in P | q \leq p\}$$

as well as

$$P_{<p} = \{q \in P | q < p\}.$$

Thus $\bar{K}_{<\sigma} = \{\tau \in \bar{K} | \tau < \sigma\} = \{\tau \in K | \tau \supset \sigma\}$, the link of σ in K.

The order complex and further decompositions:

- Given a poset P, there is an associated simplicial complex $\Delta(P)$ called the order complex of P which is defined as follows.
- Given a poset P, the *order complex* $\Delta(P)$ is the simplicial complex with vertices given by the set of points of P and k-simplices given by the ordered $(k+1)$-tuples $(p_1, p_2, \ldots, p_{k+1})$ in P with $p_1 < p_2 < \cdots < p_{k+1}$.

A further decomposition:

To state the next theorem, recall that the symbol $*$ denotes the join of two spaces.

- **Theorem 2**

  Let K be an abstract simplicial complex with m vertices, and let

  $$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$

  denote m choices of connected pairs of CW-complexes with the inclusion $A_i \subset X_i$ null-homotopic for all i. Then there is a homotopy equivalence

  $$\Sigma(Z(K; (\underline{X}, \underline{A}))) \to \Sigma \bigvee_{I} \bigvee_{\sigma \in K_I} |\Delta((\overline{K}_I))_{<\sigma}| * \widehat{D}(\sigma).$$

Examples:

- **Corollary**

  Let (X_i, A_i, x_i) denote the triple $(D^{n+1}, S^n, *)$ for all i. Then there are homotopy equivalences

$$\Sigma(Z(K;(D^{n+1}, S^n))) \to \bigvee_{I \notin K} \Sigma^{2+n|I|}|K_I|.$$

Examples continued:

- **Theorem 3**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

$$\Sigma\big(Z\big(K;(\underline{X},\underline{A})\big)\big) \rightarrow \Sigma\big(\bigvee_{I \in K} \widehat{X}^{I}\big).$$

Examples continued:

- **Theorem 4**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that the the X_i are contractible for all i. Then there is a homotopy equivalence

$$\Sigma Z(K; (\underline{X}, \underline{A})) \to \Sigma\Big(\bigvee_{I \notin K} |K_I| * \widehat{A}^I\Big).$$

Examples continued:

- **Theorem 5**

  Let K be an abstract simplicial complex with m vertices and $(\underline{X}, \underline{A})$ have the property that all the A_i are contractible. Then there is a homotopy equivalence

  $$\widehat{Z}(K; (\underline{X}, \underline{A})) = \begin{cases} * & \text{if } K \text{ is not the simplex } \Delta[m-1], \text{ and} \\ \widehat{X}^{[m]} & \text{if } K \text{ is the simplex } \Delta[m-1] \end{cases}$$

  where

  $$\widehat{X}^{[m]} = X_1 \wedge \cdots \wedge X_m$$

  is the m-fold smash product.

Applications of Theorems 3 and 5 to cohomology:

- A finite set of path-connected spaces $X_1, \ldots, X_m$ is said to satisfy the **strong form** of the Künneth Theorem over R provided that the natural map

$$\otimes_{1 \leq j \leq k} H^*(X_{i_j}; R) \to H^*(X_{i_1} \times \cdots \times X_{i_k}; R)$$

  is an isomorphism for every sequence of integers $1 \leq i_1 < i_2 < \cdots < i_k \leq m$.
- Assume throughout this section that the cohomology ring $H^*(X; R)$ satisfies the natural strong form of the Künneth theorem for the cohomology of X. Thus the natural map

$$H^*(X; R)^{\otimes m} \to H^*(X^m; R)$$

  is an isomorphism.

Applications of Theorems 3 and 5 to cohomology continued:

- With these assumptions, define the **generalized Stanley-Reisner ideal**

$$I(K) \subset H^*(X;R)^{\otimes m}$$

as the two-sided ideal generated by all elements

$$x_{j_1} \otimes x_{j_2} \otimes \cdots \otimes x_{j_r}$$

for which $x_{j_t} \in \bar{H}^*(X_{j_t};R)$ and the sequence $J = (j_1, \ldots, j_r)$ is not a simplex of K.

Applications of Theorems 3 and 5 to cohomology continued:

- Let K be an abstract simplicial complex with m vertices and let
$$(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^{m}$$
be m pointed, connected CW-pairs. Assume that all of the A_i are contractible, and coefficients are taken in a ring R for which either
  1. R is a field, or more generally
  2. the cohomology of X with coefficients in R satisfies the strong form of the Künneth Theorem.

  Then there is an isomorphism of algebras
$$(\bigotimes_{i=1}^{m} H^*(X_i; R)/I(K) \to H^*(Z(K; (\underline{X}, \underline{A})); R).$$

Applications of Theorems 3 and 5 to cohomology continued:

- This geometric decomposition then implies a decomposition of

$$E^*(Z(K;(\underline{X},\underline{A})))$$

for any cohomology theory $E^*(-)$, the starting point of the computation for KO^* of the Davis-Januszkiewicz spaces (the subject of a **nine author** paper).

- The structure arising from the suspension of the generalized moment-angle complex forces the structure of the associated cohomology rings for many of the spaces $Z(K;(X,A))$.
- However, some features of the cup-product remain unclear.

A second structure theorem for the cup-product:

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(X_i, A_i, x_i)\}_{i=1}^m$ is a family of pointed, based CW-pairs.
- Let K be an abstract simplicial complex with m vertices. Assume that the finite products

  $$(X_1 \times \cdots \times X_m) \times (Z(K_{I_1}; (D^1, S^0)) \times \cdots \times Z(K_{I_t}; (D^1, S^0)))$$

  for all $I_j \subseteq [m]$ satisfies the strong form of the Künneth theorem.
- Then the cup-product structure for the cohomology algebra $H^*(Z(K; (\underline{CX}, \underline{X})))$ is a functor of the cohomology algebras of X_i for all i, and $Z(K_I; (D^1, S^0))$ for all I.

Homework:

- Work out the cohomology ring $H^*(Z(K;(\underline{CX},\underline{X})))$.

A further structure theorem:

- Let K be an abstract simplicial complex with m vertices. Assume that $(\underline{X}, \underline{A}) = \{(A_i \vee B_i, A_i, x_i)\}_{i=1}^m$ is a family of pointed, based CW-pairs.
- Then there is a natural, pointed homotopy equivalence

$$\bigvee_{\sigma \in K} W(\sigma; \underline{A \vee B}, \underline{A})) \to \widehat{Z}(K; (\underline{A \vee B}, \underline{A}))$$

where

- $$W(\sigma; \underline{A \vee B}, \underline{A})) = Y_1 \wedge \cdots \wedge Y_m,$$

with

  - $Y_j = B_j \quad \text{if} \quad j \in \sigma$, and
  - $Y_j = A_j \quad \text{if} \quad j \in [m] - \sigma$.

- This structure gives the additive homology of $Z(K; (\underline{X}, \underline{A}))$ in case the map $H_*(\underline{A}) \to H_*(\underline{X})$ is a split monomorphism.

A language and context for legged robotic motion

- This section is based on joint work with Clark Haynes, and Dan Koditschek.

- The problem is to devise a practical, useful language for describing legged motion of certain robots.

Setting

- The topological ingredients are a space of positions again, the so-called moment-angle complexes.
- The interiors of cells in a cell decomposition gives ‘gait states’ for the legs of a legged motion.
- The purpose here is to describe the possible gait states in terms of Young diagrams, then to construct vector fields on these interiors.
- Further applications are intended.

Videos of spaces of legs:

- The following are videos of moment-angle complexes.
- The first video is of an actual direct application in joint work with Clark Haynes and Dan Koditschek.

Thank you very much !

- Please remember to hand in the homework problem to work out the cohomology algebras for $Z(K;(\underline{X},\underline{A}))$.