Cyclic structure for mappings

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Introduction

Denote by $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{D}$ the subgroups $\mathbf{I}_d = \mathbb{Z}/2$, $\mathbf{I}_a = \mathbb{Z}/4$ in the dihedral group $\mathbf{D}$. Let $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ be a map in general position, having critical points, where $k \equiv 0 \pmod 2$. Let $N^{n-2k}(d)$ be the manifold of double self-intersection points of the map d with the boundary $\partial N^{n-2k}(d)$. We say that this map d admits a cyclic structure, if there exists a mapping $\mu_{a,N(d)} : (N^{n-2k}(d), \partial N^{n-2k}(d)) \to (K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1))$, satisfying the boundary condition:

$$\mu_{a,N(d)}|_{\partial N^{n-2k}(d)} = (i_a \circ \kappa)|_{\partial N(d)}, \tag{1}$$

where $\kappa : \mathbb{RP}^{n-k} \to K(\mathbf{I}_d, 1)$ is the generating cohomology class, $i_a : K(\mathbf{I}_d, 1) \subset K(\mathbf{I}_a, 1)$ is the inclusion of a subgroup $i_{d,a} : \mathbf{I}_d \subset \mathbf{I}_a$. Moreover for an arbitrary integer q satisfying the condition $n \ge n - 2k - 2q \ge 1$, the following equation is satisfied:

$$\langle \mu^*_{a,N(d_q)}(t_q); [N^{n-2k-2q}(d_q), \partial N^{n-2k-2q}(d_q)] \rangle = 1, \tag{2}$$

where $t_q \in H^{n-2k-2q}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1); \mathbb{Z}/2)$ is an arbitrary cohomology class which is mapped to the generator of the cohomology group $H^{n-2k-2q}(K(\mathbf{I}_a, 1); \mathbb{Z}/2)$ by the induced homomorphism

$$j^* : H^{n-2k-2q}(K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1); \mathbb{Z}/2) \to H^{n-2k-2q}(K(\mathbf{I}_a, 1); \mathbb{Z}/2),$$

and $[N^{n-2k-2q}(d_q), \partial N^{n-2k-2q}(d_q)]$ is the relative fundamental class.

Main Theorem. *For $3(n-k) < 2n$, i.e. in the metastable range, there exists a generic* PL-*mapping $d_0 : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ with singularities admitting a cyclic structure.*

Remark. In the paper [Akh] this theorem was proved under weaker dimensional assumption. For applications the case $5(n-k) < 4n$ is required.

1 Auxiliary mappings

Construction of axillary mappings $c_0 : \mathbb{RP}^{n-k} \to \mathbb{R}^n$, $\hat{c}_0 : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$

Let us denote by $J_0 \subset \mathbb{R}^n$ the standard sphere of dimension $(n-k)$ in Euclidean space. This sphere is PL - homeomorphic to the join of $\frac{n-k+1}{2} = r_0$ copies of the circles S^1. Let us denote by

$$i_{J_0} : J_0 \subset \mathbb{R}^n \tag{3}$$

the standard embedding.

The mapping $p_0' : S^{n-k} \to J_0$ is well defined as the join of r_0 copies of the standard 4-sheeted coverings $S^1 \to S^1/\mathbf{i}$. The standard action $\mathbf{I}_a \times S^{n-k} \to S^{n-k}$ commutes with the mapping p_0'. Thus, the map $\hat{p}_0 : S^{n-k}/\mathbf{i} \to J_0$ is well defined and the map $p_0 : \mathbb{RP}^{n-k} \to J_0$ is well defined as the composition $\hat{p}_0 \circ \pi_0 : \mathbb{RP}^{n-k} \to J_0$ of the standard double covering $\pi_0 : \mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i}$ with the map $\hat{p}_0$.

The required mapping c_0 is denoted by the composition $i_{J_0} \circ p_0 : \mathbb{RP}^{n-k} \to J_0 \subset \mathbb{R}^n$. The required mapping $\hat{c}_0$ is denoted by the composition $i_{J_0} \circ \hat{p}_0 : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$.

2 Configuration spaces and singularities

Subspaces and factorspaces of the 2-configuration space for $\mathbb{RP}^{n-k}$, related with the axillary mapping c_0

The space Γ_0 is the 2-points configuration space of $\mathbb{RP}^{n-k}$. The subspace $\Gamma_{0\circ} \subset \Gamma_0$ is the subspace outside the diagonal of Γ_0. The structural mapping $\eta_{\Gamma_{0\circ}} : \Gamma_{0\circ} \to K(\mathbf{D}, 1)$ is well defined.

Denote by $\Sigma_{0\circ} \subset \Gamma_{0\circ}$ the polyhedron of double-points singularities of the map $p_0 : \mathbb{RP}^{n-k} \to J_0$: $\{[(x,y)] \in \Gamma_{0\circ}, p_0(x) = p_0(y), x \neq y\}$. This polyhedron is equipped with a structural mapping

$$\eta_{\Sigma_{0\circ}} : \Sigma_{0\circ} \to K(\mathbf{D}, 1), \tag{4}$$

which is induced by the restriction of the structural mapping $\eta_{\Gamma_{0\circ}}$ on the subspace $\Sigma_{0\circ}$.

Let us denote by $\Sigma_{antidiag} \subset \Gamma_{0\circ}$ a subspace, called the antidiagonal, which is formed by all antipodal pairs $\{[(x,y)] \in \Gamma_{0\circ} : x, y \in \mathbb{RP}^{n-k}, x \neq y, T_{\mathbb{RP}}(x) = y\}$.

The subpolyhedron $\Sigma_{0\circ} \subset \Gamma_{0\circ}$ of multiple-points of the map p is represented by a union $\Sigma_{0\circ} = \Sigma_{antidiag} \cup K_{0\circ}$, where $K_{0\circ}$ is an open subpolyhedron contains all points of $\Sigma_{0\circ}$ outside the antidiagonal.

Denote the closure of $Cl(K_{0\circ})$ of the polyhedron $K_{0\circ}$ in Γ_0 by K_0. Denote by $Q_{antidiag}$ the space $\Sigma_{antidiag} \cap K_0$, denote by Q_{diag} the space $\partial\Gamma_{diag} \cap K_0$. Obviously, $Q_{diag} \subset K_0$, $Q_{antidiag} \subset K_0$. We shall call these subspaces the components of the boundary of the polyhedron K_0.

Note that the structural mapping of $\eta_{K_{0\circ}}$ is extended from $K_{0\circ}$ to the component $Q_{antidiag}$ of the boundary. Denote this extension by $\eta_{Q_{antidiag}} : Q_{antidiag} \to K(\mathbf{D}, 1)$. The mapping $\eta_{Q_{antidiag}}$ is a composition $\eta_{antidiag} : Q_{antidiag} \to K(\mathbf{I}_a, 1)$ and the inclusion $i_{\mathbf{I}_a,\mathbf{D}} : K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$.

Note that the mapping η_{K_0} is not extendable to boundary component Q_{diag}. The mapping $\eta_{diag} : Q_{diag} \to K(\mathbf{I}_d, 1)$ is well defined. Let us denote by $U(Q_{diag\circ}) \subset K_{0\circ}$ a small regular deleted neighborhood of Q_{diag}. The projection $proj_{diag} : U(Q_{diag\circ}) \to Q_{diag}$ of the regular deleted neighborhood to Q_{diag}. The restriction of the structural mapping $\eta_{K_{0\circ}}$ to the neighborhood $U(Q_{diag\circ})$ is represented by a composition of the map $\eta_{U(Q_{diag\circ})} : U(Q_{diag\circ}) \to K(\mathbf{I}_b, 1)$ and the maps $i_{\mathbf{I}_b,\mathbf{D}} : K(\mathbf{I}_b, 1) \to K(\mathbf{D}, 1)$. Homotopy classes of maps $\eta_{K_0}|_{Q_{diag}}$ and $\eta_{U(Q_{diag\circ})}$ satisfy the equation:

$$\eta_{diag} \circ proj_{diag} = p_{\mathbf{I}_b,\mathbf{I}_d} \circ \eta_{U(Q_{diag\circ})}.$$

Resolution spaces for the polyhedra K_0, $\hat{K}_0$

We construct a space RK_0, which we call resolution space of the polyhedron K_0.

$$K(\mathbf{I}_a, 1) \stackrel{\phi_0}{\longleftarrow} RK_0 \stackrel{pr_0}{\longrightarrow} K_0 \qquad (5)$$

Let us introduce the following notations: $RQ_{diag} = (pr_0)^{-1}(\hat{Q}_{diag})$, $RQ_{antidiag} = (pr_0)^{-1}(Q_{antidiag})$. These spaces are included in the following commutative diagrams:

$$\begin{array}{ccc} RQ_{antidiag} & \stackrel{pr}{\longrightarrow} & Q_{antidiag} \\ \phi_0 \searrow & & \swarrow \eta_{antidiag} \\ & K(\mathbf{I}_a, 1), & \end{array} \qquad (6)$$

$$\begin{array}{ccccc} RQ_{diag} & & \stackrel{pr}{\longrightarrow} & & Q_{diag} \\ \phi_0 \searrow & & & & \swarrow \eta_{diag} \\ & & K(\mathbf{I}_d, 1). & & \end{array} \qquad (7)$$

To prove the main result of the section we need following lemma.

Lemma 1. *There is a RK_0 which is included in the commutative diagram* (5)*. Moreover, the following commutative diagrams* (6), (7)*, determine the boundary conditions.*

The beginning of the proof of Main Theorem

Consider the map $\hat{c}_0 : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$. The mapping $\hat{d}_0$ is obtained from $\hat{c}_0$ by an arbitrary C^1-small deformation in the codimension 3, this deformation is vertical with respect to the orthogonal projection $proj_{J_0}$ of a small neighborhood U_{J_0} of the embedded sphere $i_{J_0} : J_0 \subset \mathbb{R}^n$ on its central sphere $Im(i_{J_0})$. The mapping $d_0 : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ define as the result of an additional generic PL–deformation. The caliber of the deformation $\hat{d}_0 \circ \pi_0 \mapsto d_0$ is much less then the caliber of the deformation $\hat{c}_0 \mapsto \hat{d}_0$.

The following commutative diagram (8) of maps of polyhedra with described boundary conditions under the diagram is well defined. To the spaces in the third line of this diagram are mapped the spaces of the central line of the commutative diagram (9) correspondingly.

$$\begin{array}{ccc} K(\mathbf{I}_a, 1) & & \\ \uparrow \phi_0 & \nwarrow \phi_0 & \\ RK_0 & \longleftarrow & RQ_{diag} \cup RQ_{antidiag} \\ \downarrow pr_0 & & \downarrow \\ K_0 & \supset & Q_{diag} \cup Q_{antidiag} \\ \cup & & \cup \\ K_{0\circ} & \supset & \emptyset \\ \downarrow \eta_{K_0} & & \\ K(\mathbf{D}, 1). & & \end{array} \qquad (8)$$

$$
\begin{array}{ccc}
K(\mathbf{I}_a,1) & & \\
\uparrow \mu_a & & \nwarrow \mu_a \\
(N_0 \setminus N3_0)' & \supset & (N_0 \setminus N3_0)'_{diag} \cup (N_0 \setminus N3_0)'_{antidiag} \\
\cup & & \cup \\
(N_{0\circ} \setminus N3_{0\circ})' & \supset & \emptyset \\
\downarrow \eta_{K_0} & & \\
K(\mathbf{D},1). & &
\end{array} \qquad (9)
$$

In this dyagrame by

$$Nl_{0\circ} \subset N_{0\circ} \qquad (10)$$

is denoted the subpolyhedron of self-intersection points of the map $d'_0 = \pi_0 \circ \hat{d}_0 : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ of the multiplicity of l and above, where $l = 3$.

The polyhedron $N3_{0\circ} \subset N_{0\circ}$ of self-intersection points of the multiplicity greater or equal to 3 is given by the formula:

$$N3_{0\circ} = \{(x,y) \in K_{0\circ} | \exists z \in \mathbb{RP}^{n-k}, z \neq x, z \neq y : d'_0(z) = d'_0(x) = d'_0(y)\}. (11)$$

Lemma 2. *There exists a small vertical PL-deformation $\hat{c}_0 \mapsto \hat{d}_0$ such that for the subpolyhedron $(N_0 \setminus N3_0)' \subset (N_0 \setminus N3_0) \subset N_0 \subset K_0$ in the middle row of the commutative diagram (9) there exists a map $t_0 : (N_0 \setminus N3_0)' \to RK_0$, called L-resolution, to the corresponding spaces of the second row of the diagrams (8). The map $\mu_{N_0 \setminus N3_0} : (N_0 \setminus N3_0)' \to K(\mathbf{I}_a, 1)$ are defined by the formula $\mu_{N_0 \setminus N4_0} = \phi_0 \circ t_0$. This maps are uniquely extendable to the required map $\mu_a : N_0 \to K(\mathbf{I}_a, 1)$.*

Moreover, for the map μ_a the boundary condition over the components N_{diag}, $N_{antidiag}$ of the boundary are well defined and are given below the diagram (9).

3 Coordinate system angle-momentum on the spaces of singularities and construction of the resolution spaces

A preliminary step in the proof of Lemma 2

Let us present the plan of the proof. We start by an explicit description of the polyhedra K_0 and the structural maps η on these polyhedra by means of coordinates. Then we construct the spaces RK_0, equipped with maps $pr_0 : RK_0 \to K_0$ and the mapping $\phi_0 : RK_0 \to K(\mathbf{I}_a, 1)$, which satisfy required boundary conditions (5), (6).

The stratification of polyhedra J_0, K_0, $\hat{K}_0$ $K_{0\circ}$, $\hat{K}_{0\circ}$ by means of the coordinate system angle - momentum

Let us order lens spaces, which form the join, by the integers from 1 up to r_0 and let us denote by $J_0(k_1, \dots, k_s) \subset J_0$ the subjoin, formed by a selected set of circles (one-dimensional lens spaces) $S^1/\mathbf{i}$ with indexes $1 \le k_1 < \dots < k_s \le r_0$, $0 \ge s \ge r_0$. The stratification above is induced from the standard stratification of the open faces of the standard r_0-dimensional simplex δ^{r_0} under the natural projection $J_0 \to \delta^{r_0}$. The preimages of vertexes of a simplex are the lens spaces $J_0(j) \subset J_0$, $J_0(j) \approx S^1/\mathbf{i}$, $1 \le j \le r_0$, generating the join.

Define the space $J_0^{[s]}$ as a subspace of J_0, obtained by the union of all subspaces $J_0(k_1, \dots, k_s) \subset J_0$.

Denote the maximum open cell of the space $\hat{p}^{-1}(J_0(k_1, \dots, k_s))$ by $\hat{U}(k_1, \dots, k_s) \subset S^{n-k}/\mathbf{i}$. This open cell is called an elementary stratum of the depth $(r_0 - s)$. A point at an elementary stratum $U(k_1, \dots, k_s) \subset S^{n-k}/\mathbf{i}$ is defined by a set of coordinates $(\check{x}_{k_1}, \dots, \check{x}_{k_s}, l)$, where $\check{x}_{k_i}$ is a coordinate on the 1-sphere (circle), covering lens space with the number k_i, l is a coordinate on the corresponding $(s-1)$-dimensional simplex of the join. Thus if the two sets of coordinates are identified under the transformation of the cyclic $\mathbf{I}_a$-covering by means of the generator, which is common to the entire set of coordinates, then these sets define the same point on $S^{n-k}/\mathbf{i}$. Points on elementary stratum $\hat{U}(k_1, \dots, k_s)$ belong in the union of simplexes with vertexes belong to the lens spaces of the join with corresponding coordinates. Each elementary strata $\hat{U}(k_1, \dots, k_s)$ is a base space of the double covering $U(k_1, \dots, k_s) \to \hat{U}(k_1, \dots, k_s)$, which is induced from the double covering $\mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i}$ by the inclusion $\hat{U}(k_1, \dots, k_s) \subset S^{n-k}/\mathbf{i}$.

The polyhedron $K_{0\circ} = K_0 \setminus (Q_{diag} \cup Q_{antydiag})$ is slitted into the union of open subsets (elementary strata) $K_0(k_1,\dots,k_s)$, $1 \le s \le r_0$ correspondingly with the stratification

$$J_0^{(r_0)} \subset \dots \subset J_0^{(1)} \subset J_0^{(0)}, \qquad (12)$$

of the space J_0. For the considered stratum a number $r_0 - s$ of missed coordinates to the full set of coordinates is called the deep of the stratum.

Let us describe an elementary stratum $K_0(k_1,\dots,k_s)$ by means of the coordinate system. To simplify the notation let us consider the case $s = r_0$. Suppose that for a pair of points $(x_1,\ x_2)$, defining a point on $K(1,\dots,r_0)$, the following pair of points $(\check{x}_1, \check{x}_2)$ on the covering space S^{n-k} is fixed, and the pair $(\check{x}_1, \check{x}_2)$ is mapped to the pair $(x_1,\ x_2)$ by means of the projection of $S^{n-k} \to \mathbb{RP}^{n-k}$. Accordingly to the construction above, we denote by $(\check{x}_{1,i}, \check{x}_{2,i})$, $i = 1,\dots,r_0$ a set of spherical coordinates of each point. Each such coordinate with the number i defines a point on 1-dimensional sphere (circle) S^1_i with the same number i, which covers the corresponding circle $J_0(i) \subset J_0$ of the join. Note that the pair of coordinates with the common number determines the pair of points in a common layer of the standard cyclic $\mathbf{I}_a$-covering $S^1 \to S^1/\mathbf{i}$.

The collection of coordinates $(\check{x}_{1,i}, \check{x}_{2,i})$ are considered up to independent changes to the antipodal. In addition, the points in the pair (x_1, x_2) does not admit a natural order and the lift of the point in K_0 to a pair of points $(\bar{x}_1, \bar{x}_2)$ on the sphere S^{n-k}, is well determined up to 8 different possibilities. (The order of the group $\mathbf{D}_4$ is equal to 8.)

An analogous construction holds for points on deeper elementary strata $K_0(k_1,\dots,k_s)$, $1 \le s \le r_0$.

Let us reformulate the above definition of the polyhedron $\hat{K}_0^{[i]}$, $0 \le i \le r_0$. We define this polyhedron as the disjoint union of all elementary strata of the depth i. When $i \ge 1$ are considered strata, from the diagonal or the antidiagonal. (For $i = 0$ diagonal and antidiagonal striata are not considered.)

The coordinate description of elementary strata of the spaces $K_{0,\circ}$

Let $x \in K_0(1,\dots,r_0)$ be a point on a maximal elementary stratum. Consider the sets of spherical coordinates $\check{x}_{1,i}$ и $\check{x}_{2,i}$, $1 \le i \le r_0$ of the point x. For each i the following cases: a pair of i-th coordinates coincides; antipodal, the second coordinate is obtained from first by the transformation by means of the generator (or by the minus generator) of the cyclic cover. Associate to an ordered pair of coordinates $\check{x}_{1,i}$ and $\check{x}_{2,i}$, $1 \le i \le r_0$ the residue v_i of a value $+1$, -1, $+\mathbf{i}$ or $-\mathbf{i}$, respectively.

When the collection of coordinates of a point is changed to the antipodal collection, say, the collection of coordinates of the point x_2 is changed to the antipodal collection, the set of values of residues of the new pair $(\bar{x}_1, \bar{x}_2)$ on the spherical covering is obtained from the initial set of residues by changing of the signs. The residues of the renumbered pair of points change by the inversion. Obviously, the set of residues does not change, if we choose another point on the same elementary stratum of the space $K_{0\circ}$.

Elementary strata of the space $K_0(1, \dots, r_0)$, in accordance with sets of residues, are divided into 3 types: $\mathbf{I}_a, \mathbf{I}_b, \mathbf{I}_d$. If among the set of residues are only residues $\{+\mathbf{i}, -\mathbf{i}\}$ (respectively, only residues $\{+1, -1\}$), we shall speak about the elementary stratum of the type $\mathbf{I}_a$ (respectively of the type $\mathbf{I}_b$). If among the residues are residues from the both set $\{+\mathbf{i}, -\mathbf{i}\}$ and $\{+1, -1\}$, we shall speak about elementary stratum of the type $\mathbf{I}_d$. It is easy to verify that the restriction of the structure mapping $\eta : K_{0\circ} \to K(\mathbf{D}_4, 1)$ on an elementary stratum of the type $\mathbf{I}_a, \mathbf{I}_b, \mathbf{I}_d$ is represented by the composition of a map in the space $K(\mathbf{I}_a, 1)$ (respectively in the space $K(\mathbf{I}_b, 1)$ or $K(\mathbf{I}_d, 1)$) with the map $i_a : K(\mathbf{I}_a, 1) \to K(\mathbf{D}_4, 1)$ (respectively, with the map $i_b : K(\mathbf{I}_b, 1) \to K(\mathbf{D}_4, 1)$ or $i_d : K(\mathbf{I}_d, 1) \to K(\mathbf{D}_4, 1)$). For the first two types of strata the reduction of the structural mapping (up to homotopy) is not well defined, but is defined only up to a composition with the conjugation in the subgroups $\mathbf{I}_a$, $\mathbf{I}_b$.

Description of the structural map $\eta_\circ : K_{0\circ} \to K(\mathbf{D}, 1)$, by means of the coordinate system

Let $x = [(x_1, x_2)]$ be a marked a point on $K_(0, \circ)$, on a maximal elementary stratum. Consider closed path $\lambda : S^1 \to K_{0,\circ}$, with the initial and ending points in this marked point, intersecting the singular strata of the depth 1 in a general position in a finite set of points. Let $(\check{x}_1, \check{x}_2)$ be the two spherical preimages of the point x. Define another pair $(\check{x}'_1, \check{x}'_2)$ of spherical preimages of x, which will be called coordinates, obtained in result of the natural transformation of the coordinates $(\check{x}_1, \check{x}_2)$ along the path λ.

At regular points of the path λ the family of pairs of spherical preimages in the one-parameter family is changing continuously, that uniquely identifies the inverse images of the end point of the path by the initial data. When crossing the path with the strata of depth 1, the corresponding pair of spherical coordinates with the number l is discontinuous. Since all the other coordinates remain regular, the extension of regular coordinates along the path at a critical moment time is uniquely determined. For a given point x on elementary stratum of the depth 0 of the spaces $K_{0,\circ}$ the choice of at least one pair of spherical coordinates is uniquely determines the choice of spherical coordinates with the rest numbers. Consequently, the continuation of the spherical coordinates along a path is uniquely defined in a neighborhood of a singular point of the path.

The transformation of the ordered pair $(\check{x}_1, \check{x}_2)$ to the ordered pair $(\check{x}'_1, \check{x}'_2)$ defines an element the group $\mathbf{D}$. This element does not depend on the choice of the path l in the class of equivalent paths, modulo homotopy relation in the group $\pi_1(K_{0\circ}, x)$. Thus, the homomorphism $\pi_1(K_{0\circ}, x) \to \mathbf{D}$ is well defined and the induced map

$$\eta_\circ : K_{0\circ} \to K(\mathbf{D}, 1) \tag{13}$$

coincides with structural mapping, which was determined earlier. It is easy to verify that the restriction of the structural mapping η on the connected components of a single elementary stratum $K_{0\circ}(1, \dots, r)$ is homotopic to a map with the image in the subspeces $K(\mathbf{I}_a, 1)$, $K(\mathbf{I}_b, 1)$, $K(\mathbf{I}_d, 1)$, which corresponds to the type and subtype elementary stratum.

Consider an elementary stratum $K_0(k_1, \dots, k_s) \subset K_{0\circ}^{(r_0-s)}$ of the depth of $(r_0 - s)$. Denote by

$$\pi : K_0(k_1, \dots, k_s) \to K(\mathbb{Z}/2, 1) \tag{14}$$

the classifying map, that is responsible for the permutation of a pair of points around a closed path on this elementary stratum.

The mapping π coincides with the composition

$$K_0(k_1, \dots, k_s) \stackrel{\eta}{\longrightarrow} K(\mathbf{D}_4, 1) \stackrel{p}{\longrightarrow} K(\mathbb{Z}/2, 1),$$

where $K(\mathbf{D}_4, 1) \stackrel{p}{\longrightarrow} K(\mathbb{Z}/2, 1)$ be the map of the classifying spaces, which is induced by the epimorphism $\mathbf{D}_4 \to \mathbb{Z}/2$ with kernel $\mathbf{I}_c \subset \mathbf{D}_4$.

References

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