Cyclic structure for mappings

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P.M. Akhmet'ev, IZMIRAN

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Introduction

In the paper [Akh1] a definition of cyclic structure for self-transversal PL–mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$, with, generally speaking, critical points was proposed. Assuming the dimensional restriction

$$n - 4k \geq 7, \quad n = 2^\ell - 1, \quad n - k \equiv 0 \pmod 4, \tag{1}$$

this definition was used in [A1] to investigate problems in stable homotopy groups of spheres. In the paper [Akh1] the notion of cyclic structure was considered only in the case $n - 2k \geq 15$. In this paper we present another definition of cyclic structure

In the first part of the paper we investigate the dimensional restriction

$$n - 3k \geq -8, \quad n - k \equiv 0 \pmod 2. \tag{2}$$

And prove Lemma $2A$. In the second part we prove the main result: Lemma $2B$, assuming the following dimensional restriction

$$n - 5k \geq -16, \quad n - k \equiv 0 \pmod 4. \tag{3}$$

By the dimensional restriction (3) the condition (1) is possible only in the case $n \ge 127$, i.e. in the case $\ell \ge 7$. This case is used in the Main Result in [A1].

Let us start by the main definition. Denote by $\mathbf{I}_d \subset \mathbf{I}_a \subset \mathbf{D}$ the following subgroups $\mathbf{I}_d = \mathbb{Z}/2$, $\mathbf{I}_a = \mathbb{Z}/4$ in the dihedral group $\mathbf{D}$ of the order 8 (see more details in [A1, p.5]).

Let $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ be an arbitrary PL-mapping. Denote by $N = N(d)$ the polyhedron of self-intersection points of the mapping d (see the formula (18) below). This polyhedron, generally speaking, has a boundary ∂N (this boundary consists of critical points of the mapping d). Assuming the condition (2), a self-transversal mapping can have only self-intersection points of the multiplicity 2 and the embedding $N \subset \mathbb{R}^n$ is well defined. In the considered case the polyhedron N has the dimension $n-2k$. Denote by $N_\circ$ an open polyhedron $N \setminus \partial N$, denote by $U(\partial N)_\circ$ a deleted regular neighborhood of the boundary ∂N.

For an arbitrary $N = N(d)$ the canonical 2–sheeted covering $\bar{N} \to N$ with ramification over the boundary ∂N is well defined (see the formula (19). The standard inclusion $i_{\bar{N}} : \bar{N} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ is well defined. The inverse image of the diagonal $\mathbb{RP}^{n-k}_{diag} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ with respect to the inclusion $i_{\bar{N}}$ coincides with the boundary ∂N. The inclusion $i_{\bar{N}}$ is invariant with respect to the standard involutions (given by the permutation of the preimages) in the target and in the source, denoted by $T_{\bar{N}}$ и $T_{\mathbb{RP}^{n-k}}$ correspondingly.

Let us assume that the image of the mapping d is contained on the surface of the standard embedding sphere $S^{n-k} \subset \mathbb{R}^n$. In this case the following $T_{\bar{N}}$-, $T_{\mathbb{RP}^{n-k}}$-, $T_{S^{n-2}}$-, and $T_{\mathbb{R}^n}$-equivariant mappings (the first and the third equivariant mappings are equivariant embeddings):

$$\bar{N} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to S^{n-k} \times S^{n-k} \subset \mathbb{R}^n \times \mathbb{R}^n, \qquad (4)$$

where by $T_{S^{n-k}}$ is denoted the standard involution on $S^{n-k} \times S^{n-k}$, by $T_{\mathbb{R}^n}$ is denoted the standard involution on $\mathbb{R}^n \times \mathbb{R}^n$. The inverse image of the diagonal $S^{n-k}_{diag} \subset S^{n-k} \times S^{n-k}$ contains $\mathbb{RP}^{n-k}_{diag}$, the inverse image of the diagonal $\mathbb{R}^n_{diag} \subset \mathbb{R}^n \times \mathbb{R}^n$ coincides to S^{n-k}_{diag}.

Let $d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n$ be an arbitrary $T_{\mathbb{RP}^{n-k}}$, $T_{\mathbb{R}^n}$–equivariant mapping, transversal along the diagonal. Denote $(d^{(2)})^{-1}(\mathbb{R}^n_{diag})/T_{\mathbb{RP}^{n-k}}$ by $N = N(d^{(2)})$ and let us call this polyhedron the polyhedron of (formal) intersection of the mapping $d^{(2)}$. For an arbitrary point $x = (x_1, x_2) \in N$, $x_1 \ne x_2$, denote by $U(x)$ a neighborhood of the point x, which is a Cartesian product of neighborhoods $x_1 \in V(x_1) \subset \mathbb{RP}^{n-k}$ и $x_2 \in V(x_2) \subset \mathbb{RP}^{n-k}$.

Definition 1. Let us call that an equivariant mapping $d^{(2)}$ has a holonomic (formal) self-intersection, if there exists a mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$, such that $d^{(2)}$ is the holonomic extension of d in a small regular equivariant neighborhood $U_{diag} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ of the diagonal $\mathbb{RP}^{n-k}_{diag} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$, and, moreover, if for an arbitrary point $x = (x_1, x_2) \in \bar{N} \setminus (\bar{N} \cap \bar{U}_{diag})$ the mapping $d^{(2)}$ in a small neighborhood $U(x) = V(x_1) \times V(x_2)$ of this point is a Cartesian product of the two mappings $f_1 : V(x_1) \to \mathbb{R}^n$ and $f_2 : V(x_2) \to \mathbb{R}^n$, $d^{(2)} = f_1 \times f_2$ (generally speaking, $f_i \ne d|_{V(x_i)}, i = 1, 2$).

Definition of cyclic structure (comp. with [Definition 24,A1])

Let a polyhedron $N_\circ$ be a polyhedron of (formal) self-intersection of an equivariant mapping $d^{(2)}$, $n = 2^l - 1, l \ge 7$. For an arbitrary non-negative integer q, assuming the dimensional restriction $n - 2k - 2q \ge 1$, let us define the following relative homology class

$$[N_q, \partial] \in H_{n-2k-2q}(N_\circ, U(\partial N)_\circ; \mathbb{Z}/2), \tag{5}$$

where by $U(\partial N)_\circ \subset N_\circ$ is denoted a regular deleted neighborhood of the boundary.

Denote by

$$d^{(2)}_q : \mathbb{RP}^{n-k-q} \times \mathbb{RP}^{n-k-q} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \tag{6}$$

the restriction of the equivariant mapping $d^{(2)}$ on the Cartesian product of the standard projective subspace of the codimension q. In particular, for $q = 0$ we get $d^{(2)}_q = d^{(2)}$. Let us assume that the equivariant mapping $d^{(2)}$ is generic and that each mapping (6) is also generic. Let us denote the polyhedron of (formal) self-intersection points of the mapping (6) by

$$N_q, \quad N_{q\circ} = N_q \setminus \partial N_q. \tag{7}$$

Obviously, we have $\dim(N_q) = n - 2k - 2q$.

The following standard inclusion

$$i_{N_{q\circ}} : N_{q\circ} \subset N_\circ. \tag{8}$$

is well defined.

Let us define a relative homology class (5) as the image of the relative fundamental class of the polyhedron with boundary (7) (assuming $d^{(2)}$ is generic, this relative fundamental class is well defined) by the inclusion (8).

Assume that the mapping

$$\mu_a : (N, \partial N) \to (K(\mathbf{I}_a, 1), K(\mathbf{I}_d, 1)) \tag{9}$$

is well defined.

For an arbitrary $q \geq 0$ the following characteristic number in the left side of the equation

$$\langle \mu_a^*(t_q); [N_q, \partial] \rangle = 1, \qquad (10)$$

is well defined, where $t_q \in H^{n-2k-2q}(K(\mathbf{I}_a,1), K(\mathbf{I}_d,1); \mathbb{Z}/2)$ is the generic cohomology class, which transforms to the generic cohomology class in $H^{n-2k-2q}(K(\mathbf{I}_a,1); \mathbb{Z}/2)$ by means of the homomorphism

$$j^* : H^{n-2k-2q}(K(\mathbf{I}_a,1), K(\mathbf{I}_d,1); \mathbb{Z}/2) \to H^{n-2k-2q}(K(\mathbf{I}_a,1); \mathbb{Z}/2).$$

The mapping (9) is called the cyclic structure of the equivariant mapping $d^{(2)}$ with holonomic self-intersection (in particular, if $d^{(2)}$ is a extension of a self-transversal mapping d), if the family of equations (10) are satisfied and the following boundary condition:

$$\mu_a|_{U(\partial N)_\circ} = \kappa_\circ|_{U(\partial N)_\circ}, \qquad (11)$$

is satisfied, where $\kappa_\circ : U(\partial N)_\circ \to K(\mathbf{I}_b, 1)$ is the mapping, which determines the reduction of the structural mapping $\eta_\circ$ on the deleted neighborhood of the boundary $U(\partial N)_\circ$.

The main result is the following lemma.

Lemma 2.

A. Assuming the dimensional restriction (2)*, there exists a generic PL-mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ (with singularity), such that the formal extension $d^{(2)}$ of d admits a cyclic structure. Moreover, the structured mapping $\eta_\circ : N_\circ \to K(\mathbf{D}, 1)$ admits a reduction to a mapping into $K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$, which is defined by the mapping $\mu_a : N_\circ \to K(\mathbf{I}_a, 1)$, namely, $\eta_\circ = i_{\mathbf{I}_a,\mathbf{D}} \circ \mu_a$, where $i_{\mathbf{I}_a,\mathbf{D}} : K(\mathbf{I}_a, 1) \to K(\mathbf{D}, 1)$ is the mapping, which is induced by the inclusion of the cyclic subgroup.*

B. Assuming the dimensional restriction (3)*, there exists an equivariant generic mapping $d^{(2)}$ with holonomic self-intersection, which admits a cyclic structure.*

1 Auxiliary mappings

Construction of an axillary mapping $c : \mathbb{RP}^{n-k} \to \mathbb{R}^n, \hat{c} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$

Denote by J the standard $(n-k)$–dimensional sphere, which is represented as the join of $\frac{n-k+1}{2} = r$ copies of the circle S^1. We denote the standard embedding of J into $\mathbb{R}^n$ of the codimension $n-k$ by

$$i_J : J \subset \mathbb{R}^n. \tag{12}$$

The mapping $p' : S^{n-k} \to J$ is well defined as the join of r copies of the standard 4-sheeted coverings $S^1 \to S^1/\mathbf{i}$. The standard action $\mathbf{I}_a \times S^{n-k} \to S^{n-k}$ commutes with the mapping p'. Thus, the map

$$\hat{p} : S^{n-k}/\mathbf{i} \to J \tag{13}$$

is well defined and the map (covering with ramification)

$$p = \hat{p} \circ \pi : \mathbb{RP}^{n-k} \to J \tag{14}$$

is well defined as the composition $\hat{p} \circ \pi : \mathbb{RP}^{n-k} \to J$ of the standard double covering $\pi : \mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i}$ with the map $\hat{p}$.

The required auxiliary mapping c is denoted by the composition

$$c = i_J \circ p : \mathbb{RP}^{n-k} \to J \subset \mathbb{R}^n. \tag{15}$$

The required auxiliary mapping $\hat{c}$ is denoted by the composition

$$i_J \circ \hat{p} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n. \tag{16}$$

2 Configuration spaces and singularities

Subspaces and factorspaces of the 2-configuration space for $\mathbb{RP}^{n-k}$, related with the axillary mappings c, $\hat{c}$

Let us define a manifold with boundary Γ, the interior of this manifold with boundary $\Gamma_\circ \subset \Gamma$ and 2-sheeted covering $\bar{\Gamma}$. Let us consider the configuration space

$$\Gamma = (\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Delta_{\mathbb{RP}^{n-k}})/\bar{T} \tag{17}$$

of the space $\mathbb{RP}^{n-k}$, which is called the "‘deleted product"’. This space is the quotient of the deleted Cartesian product with respect to the involution $T' : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$, which permutes the coordinates. The space (17) is an open manifold.

Define the space Γ' (a manifold with boundary) as the spherical bow-up of the space (17) along the diagonal. Recall, that the spherical blow-up is defined as the standard compactification of the open manifold $\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Delta_{\mathbb{RP}^{n-k}}$ by means of fiberwise gluing of the fibers $ST\Delta_{\mathbb{RP}^{n-k}}$ of the spherization of the tangent bundle $T\Delta_{\mathbb{RP}^{n-k}}$ of the normal bundle over the diagonal $\Delta_{\mathbb{RP}^{n-k}} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$. The following natural inclusions are well-defined:

$$\mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \setminus \Delta_{\mathbb{RP}^{n-k}} \subset \bar{\Gamma},$$

$$ST\Delta_{\mathbb{RP}^{n-k}} \subset \bar{\Gamma}.$$

On the total space of 2-sheeted canonical covering $\bar{\Gamma}'$ of the space Γ' the following free involution $\bar{T}' : \bar{\Gamma} \to \bar{\Gamma}$, which is the extension of the involution $\bar{T}$ is well defined.

The quotient $\bar{\Gamma}'/\bar{T}'$ denote by Γ', the corresponding 2-sheeted covering denote by

$$p_{\Gamma'} : \bar{\Gamma}'/\bar{T}' \to \Gamma'.$$

The space Γ' is an open manifold with boundary, this manifold is called the blow-up of the configuration space (17). The projection $p_{\partial\Gamma'} : \partial\Gamma' \to \mathbb{RP}^{n-k}$ is well defined, this projection is called the resolution of the diagonal.

For an arbitrary PL–mapping d the polyhedron $N_\circ = N_\circ(d)$ of self-intersection points of d is defined by the formula:

$$N_\circ = Cl\{([x,y]) \in int(\Gamma) : y \neq x, d(y) = d(x)\}. \qquad (18)$$

By the Porteous Theorem [Por], assuming that the mapping d is smooth and generic, the polyhedron $N_\circ$ is the interior of the manifold with boundary of the dimension $(n-2k)$, denote this manifold with boundary by $N^{n-2k}(d)$.

It is easy to check that the formula (18) determines an inclusion of the pair of polyhedra:

$$i_N : (N, \partial N) \subset (\Gamma, \partial\Gamma).$$

The boundary ∂N of the polyhedron N is called the polyhedron of critical points of the mapping d. The natural mapping $(N^{n-2k}(d), \partial N^{n-2k}(d)) \to (N, \partial N)$, called the resolution mapping, is well defined. The restriction of this mapping on $N^{n-2k}(d) \setminus \partial N^{n-2k}(d)$ is a PL-homeomorphism onto $N_\circ$. The canonical 2-sheeted covering

$$p_N : \bar{N} \to N, \qquad (19)$$

with ramification over the boundary ∂N (over this boundary the covering is a PL-homeomorphism) is well defined. The following diagram is commutative:

$$\begin{array}{ccc} i_{\bar{N}} : (\bar{N}^{n-2k}(d), \partial\bar{N}^{n-2k}(d)) & \subset & (\bar{\Gamma}', \partial\bar{\Gamma}') \\ \downarrow p_N & & \downarrow p_\Gamma \\ i_{N(d)} : (N, \partial N) & \subset & (\Gamma, \partial\Gamma). \end{array}$$

Structural mapping $\eta_{N_\circ} : N_\circ \to K(\mathbf{D}, 1)$

Let us define a mapping

$$\eta_{\Gamma'} : \Gamma' \to K(\mathbf{D}, 1), \qquad (20)$$

which is called the structural mapping of the "‘deleted product"’. Let us note that the inclusion $\bar{\Gamma} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ induced the isomorphism of the fundamental groups, because the codimension of the diagonal $\Delta_{\mathbb{RP}^{n-k}} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ satisfies the inequality $n-k \ge 3$. Therefore the following equation is satisfied:

$$\pi_1(\bar{\Gamma}) = H_1(\bar{\Gamma}; \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \qquad (21)$$

Let us consider the induced automorphism $T'_* : H_1(\bar{\Gamma}'; \mathbb{Z}/2) \to H_1(\bar{\Gamma}'; \mathbb{Z}/2)$. Note that this automorphism is not the identity. Let us fix the isomorphism $H_1(\bar{\Gamma}'; \mathbb{Z}/2)$ и $\mathbf{I}_c$, where the generator of the first (correspondingly, the second) factor of the group $H_1(\bar{\Gamma}'; \mathbb{Z}/2)$ (see (21)) is mapped to the generator $ab \in \mathbf{I}_c \subset \mathbf{D}$ (correspondingly, to the generator $ba \in \mathbf{I}_c \subset \mathbf{D}$), which in the standard representation of the group $\mathbf{D}$ is determined by the symmetry with respect to the second (correspondingly, to the first) coordinate axis.

It is easy to verify that the automorphism of the conjugation with respect to the subgroup $\mathbf{I}_c \subset \mathbf{D}$ by means of the element $b \in \mathbf{D} \setminus \mathbf{I}_c$ (in this formula the element b can be chosen arbitrarily), defined by the formula $x \mapsto bxb^{-1}$, corresponds to the automorphism T'_*. The fundamental group $\pi_1(\Gamma')$ is a quadratic extension of $\pi_1(\bar{\Gamma}')$ by means of the element b, and this extension is uniquely defined up to isomorphism by the automorphism T'_*. Therefore $\pi_1(\Gamma') \simeq \mathbf{D}$, and hence the mapping $\eta_{\Gamma'} : \Gamma' \to K(\mathbf{D}, 1)$ is well defined.

It is easy to verify that the mapping $\eta_{\Gamma'}|_{\partial\Gamma'}$ takes values in the subspace $K(\mathbf{I}_b, 1) \subset K(\mathbf{D}, 1)$. The mapping $\eta_{\Gamma'}$ induces the map

$$\eta_\circ : (N_\circ, U(\partial N)_\circ) \to (K(\mathbf{D}, 1), K(\mathbf{I}_b, 1)), \qquad (22)$$

which we call the structure mapping. (The notion of the structure mapping is analogous to the notion of the classifying mapping for $\mathbf{D}$–framed immersion.) Also, it is easy to verify that the homotopy class of the composition $U(\partial N)_\circ \xrightarrow{\eta_\circ} K(\mathbf{I}_b, 1) \xrightarrow{p_b} K(\mathbf{I}_d, 1)$ coincides with the characteristic map $\kappa : U(\partial N)_\circ \to \mathbb{RP}^{n-k} \to K(\mathbf{I}_d, 1)$, which is the composition of the resolution map $U(\partial N)_\circ \to \partial N \subset \mathbb{RP}^{n-k}$ and the embedding of the skeleton $\mathbb{RP}^{n-k} \subset K(\mathbf{I}_d, 1)$ in the classifying space.

Structural mapping $\eta_{\Sigma_\circ} : \Sigma_\circ \to K(\mathbf{D},1)$

The formula (18) is well-defined for an arbitrary mapping. Let us apply this formula to the mapping p, given by the formula (14).

Denote by $\Sigma_\circ \subset \Gamma_\circ$ the polyhedron of self-intersection points of the mapping $p : \mathbb{RP}^{n-k} \to J$, this polyhedron is defined by the formula (18) in the case $d = p$. This polyhedron is equipped with the structural mapping

$$\eta_{\Sigma_\circ} : \Sigma_\circ \to K(\mathbf{D},1), \tag{23}$$

which is induced by the restriction of the structural mapping $\eta_{\Gamma_\circ}$ on the space $\Sigma_\circ$.

Denote by

$$\Sigma_{antidiag} \subset \Gamma_\circ \tag{24}$$

the subspace, called the antidiagonal, which is defined by $\{[(x,y)] \in \Gamma_\circ : x,y \in \mathbb{RP}^{n-k}, x \ne y, T_{\mathbb{RP}}(x) = y\}$. It is not hard to check that the antidiagonal $\Sigma_{antidiag} \subset \Gamma_\circ$ is defined as the fixed point set of the involution $T_{\Gamma_\circ}$.

The polyhedron $\Sigma_\circ \subset \Gamma_\circ$ of self-intersection points of the mapping p is represented by the union

$$\Sigma_\circ = \Sigma_{antidiag} \cup K_\circ, \tag{25}$$

where $K_\circ$ is the open polyhedron, which contains all points in $\Sigma_\circ$, but points on the antidiagonal. The restriction of the structured mapping $\eta_{\Gamma_\circ} : \Gamma_\circ \to K(\mathbf{D},1)$ on $\Gamma_{K_\circ}$ and on $K_\circ$ denote by $\eta_{\Gamma_\circ}$ and $\eta_{K_\circ}$ correspondingly.

Let us consider the closure $Cl(K_\circ) \subset \Gamma$ of the polyhedron $K_\circ \subset \Gamma_\circ$, denote this polyhedron by K Denote by $Q_{antidiag}$ the space $\Sigma_{antidiag} \cap K$, denote by Q_{diag} the space $\partial\Gamma_{diag} \cap K$. Evidently, $Q_{diag} \subset K$ and $Q_{antidiag} \subset K$. Let us called the considered subspaces the boundary components of the polyhedron K.

Denote by $\eta_\circ$ the restriction of the structural mapping $\eta_{\Gamma_\circ}$ on the open polyhedron $K_\circ$. Note that the structural mapping $\eta_\circ$ extends from $K_\circ$ on the component of the boundary $Q_{antidiag}$. Denote this extension by $\eta_{Q_{antidiag}} : Q_{antidiag} \to K(\mathbf{D},1)$. The mapping $\eta_{Q_{antidiag}}$ is represented by the composition of the mapping $\eta_{antidiag} : Q_{antidiag} \to K(\mathbf{I}_a,1)$ and the standard inclusion $i_{\mathbf{I}_a,\mathbf{D}} : K(\mathbf{I}_a,1) \subset K(\mathbf{D},1)$.

The structural mapping $\eta_\circ$ does not extended to the component Q_{diag} of the boundary. The mapping

$$\eta_{diag} : Q_{diag} \to K(\mathbf{I}_d,1) \tag{26}$$

is well defined.

Let us denote by $U(Q_{diag})_\circ \subset K_\circ$ a small regular deleted neighborhood of Q_{diag}. The standard projection $proj_{diag} : U(Q_{diag})_\circ \to Q_{diag}$ is well defined. The restriction of the structural mapping $\eta_\circ$ on the neighborhood $U(Q_{diag})_\circ$ is represented by the composition of the mapping $\eta_{UQ_{diag\circ}} : U(Q_{diag})_\circ \to K(\mathbf{I}_b, 1)$ and the mapping $i_{\mathbf{I}_b,\mathbf{D}} : K(\mathbf{I}_b, 1) \to K(\mathbf{D}, 1)$. The homotopy classes of the mappings η_{diag} and $\eta_{UQ_{diag\circ}}$ are related by the following equation:

$$\eta_{diag} \circ proj_{diag} = p_{\mathbf{I}_b,\mathbf{I}_d} \circ \eta_{UQ_{diag\circ}}. \qquad (27)$$

Resolution space $R\Sigma$ of the polyhedron Σ

Below the space $R\Sigma$, which is called *resolution space* of the polyhedron Σ, is defined.

The space $R\Sigma$ is decomposed as following:

$$R\Sigma = RN \cup RL, \qquad (28)$$

where RN and RL are closed polyhedra.

The projection

$$R\Sigma \stackrel{pr}{\longrightarrow} \Sigma, \qquad (29)$$

is well defined, moreover, there is an inclusion $Q_{diag} \subset pr(RL)$.

The following mapping of triads are well defined:

$$(RN, RL; RN \cap RL) \stackrel{\phi_{RN},\phi_{RL}}{\longrightarrow} (K(\mathbf{I}_a, 1), K(\mathbf{I}_b, 1); K(\mathbf{I}_d, 1)). \qquad (30)$$

Denote by

$$\phi : R\Sigma \to K(\mathbf{I}_a, 1) \qquad (31)$$

the mapping, which coincides on the subpolyhedron $RN \subset R\Sigma$ to the mapping ϕ_{RN} from the diagram (30), and on the subpolyhedron $RN \subset R\Sigma$ coincides to the composition of the mapping ϕ_{RL} from the diagram (30) with the projection $p_{\mathbf{I}_b,\mathbf{I}_d} : K(\mathbf{I}_b, 1) \to K(\mathbf{I}_d, 1)$ (this two restrictions define the mapping ϕ).

Let us introduces the following denotation: $RQ_{diag} = (pr)^{-1}(Q_{diag})$. Denote by $U(RQ_{diag})$ the regular neighborhood and by $U(RQ_{diag})_\circ$ the regular deleted neighborhood of the subpolyhedron $RQ_{diag} \subset RK$. The neighborhood $U(RQ_{diag})$ is small, such the inclusion $U(RQ_{diag}) \subset RL$ is well defined.

The following diagram of mappings is well defined:

$$U(RQ_{diag})_{\circ} \xrightarrow{pr} U(Q_{diag})_{\circ}$$
$$\phi_{\mathbf{I}_b} \searrow \qquad \swarrow \eta_{UQ_{diag\circ}} \qquad (32)$$
$$K(\mathbf{I}_b, 1),$$

in this diagram by $\phi_{\mathbf{I}_d}$ is denoted the restriction of the mapping ϕ_{RL} (see. (30)) on the subspace $U(RQ_{diag})_{\circ} \subset RL$, the mapping $\eta_{UQ_{diag\circ}}$ is defined above the formula (27).

To prove the main result the following lemma is required.

Lemma 3. *There is a space $R\Sigma$ which is equipped with the mapping* (31)*. The commutative diagram* (32) *determines the boundary conditions.*

3 The beginning of the proof of main result in the case of dimensional restriction (2)

Let us recall that the polyhedron J is PL–homeomorphic to the standard sphere S^{n-k}. Let us consider the embedding (12) and let us present this embedding by the composition of the standard embeddings: $i_0 : J \subset \mathbb{R}^{n-k+1}$, $i_1 : \mathbb{R}^{n-k+1} \subset \mathbb{R}^{n-3}$, $i_2 : \mathbb{R}^{n-3} \subset \mathbb{R}^n$.

Let us consider the mapping $\hat{c} : S^{n-k}/\mathbf{i} \to \mathbb{R}^n$, which is given by the formula (16). Let us present this mapping as the composition of the mapping $\hat{c}_0 : S^{n-k}/\mathbf{i} \to \mathbb{R}^{n-k+1}$, the mapping $i_1 : \mathbb{R}^{n-k+1} \subset \mathbb{R}^{n-3}$ and the mapping $i_2 : \mathbb{R}^{n-3} \subset \mathbb{R}^n$.

Let us consider the mapping $\hat{c}'_1 : S^{n-k}/\mathbf{i} \to \mathbb{R}^{n-3}$, which is obtained from $\hat{c}_1 = i_1 \circ \hat{c}$ by means of C^1–small PL–deformation, which is vertical with respect to the orthogonal projection $proj_J$ of a small regular neighborhood U_J of the embedding sphere $i_1 \circ i_0 : J \subset \mathbb{R}^{n-3}$ onto the central sphere $Im(i_1 \circ i_0)$. Let us consider the mapping $p \circ \hat{c}'_1 : \mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i} \to \mathbb{R}^{n-3}$ and define the mapping $c'_1 : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-3}$ as the result of an additional $proj_J$-vertical C^1–small PL-deformation, much smaller then the deformation $i_1 \circ \hat{c}_0 \mapsto \hat{c}'_1$.

Let us denote the polyhedron of self-intersection points of the mapping c'_1 and its interior by

$$N'_{\circ} \subset N'. \qquad (33)$$

By dimensional reason the mapping c'_1 has no self-intersection points of the multiplicity 3 and greater. The codimension $codim(\Sigma(c'_1))$ of this polyhedron inside the source manifold $\mathbb{RP}^{n-k}$ is equal to $k-3$ and, assuming the dimensional restriction (2), we get: $2codim(N') > n-k$.

Because the deformation $p \circ \hat{c}_1 \mapsto c'_1$ is vertical, the polyhedron $N'_\circ$ is a subpolyhedron in $\Sigma_\circ$.

The following commutative diagram is well defined:

$$\begin{array}{ccc} N'_\circ & \supset & U(N'_{diag})_\circ \\ \downarrow \eta_\circ & & \\ K(\mathbf{D},1). & & \end{array} \qquad (34)$$

Below in Lemma 6 we define the required mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ by means of an additional deformation $i_2 \circ c'_1 \mapsto d$. The deformation $i_2 \circ c'_1 \mapsto d$, generally speaking, is not vertical with respect to $proj_J \circ (\mathbb{R}^n \to \mathbb{R}^{n-3})$, where $\mathbb{R}^n \to \mathbb{R}^{n-3}$ is the standard orthogonal projection. Properties of the mapping d are described below in Lemma 4.

The following commutative diagram (35) of maps of polyhedra with described boundary conditions under the diagram is well defined. To the spaces in the lines 4,5,6 of this diagram are mapped the spaces of the lines 2,3,4 of the commutative diagram (36) correspondingly.

$$\begin{array}{ccc} K(\mathbf{I}_a,1) & & \\ \uparrow \phi & \nwarrow \phi & \\ R\Sigma & \supset & U(RQ_{diag})_\circ \\ \downarrow pr & & \downarrow \\ K & \supset & U(Q_{diag})_\circ \\ \cup & & \cup \\ N'_\circ & \supset & U(N'_{diag})_\circ \\ \downarrow \eta_\circ & & \\ K(\mathbf{D},1). & & \end{array} \qquad (35)$$

$$
\begin{array}{ccc}
K(\mathbf{I}_a,1) & & \\
\uparrow \mu_a & & \nwarrow \mu_a \\
N_\circ & \supset & U(N_{diag})_\circ \\
\downarrow \eta_\circ & & \\
K(\mathbf{D},1). & &
\end{array}
\tag{36}
$$

Boundary conditions in the deleted neighborhood $U(N_{diag})_\circ$ are given by the formula: $\mu_a = \eta_{diag} : U(N_{diag})_\circ \longrightarrow K(\mathbf{I}_d,1) \longrightarrow K(\mathbf{I}_a,1)$.

In this diagram (36) by $N_\circ$ is denoted an open polyhedron of self-intersection points of the mapping d. Denote by N the closure of the polyhedron $N_\circ$. Denote by $U(N_{diag})_\circ$ a small regular deleted neighborhood of the diagonal N_{diag} of the polyhedron N.

Lemma 4. *There exists a C^0–small PL-deformation $i_2 \circ c'_1 \mapsto d$ such that for the polyhedron N there exists a resolution map $rez : N \to R\Sigma$ to the corresponding space of the second row of the diagrams* (35). *The map $\mu_a : N \to K(\mathbf{I}_a,1)$ (see the equation* (9)*) is defined by the formula $\mu_a = \phi \circ rez$, where the mapping ϕ is defined by the formula* (31).

The restriction of μ_a on $N_\circ \subset N$ determines the reduction of the structural mapping: $i_{\mathbf{I}_a,\mathbf{D}} \circ \mu_a = \eta_\circ : N_\circ \to K(\mathbf{D},1)$ (see the formula (22)*).*

In particular, the mapping μ_a satisfies the required boundary conditions over N_{diag} and induces a cyclic structure of the extension $d^{(2)}$.

4 Coordinate system angle-momentum on the spaces of singularities and construction of the resolution spaces

A preliminary step in the proof of Lemma 4

Let us present the plan of the proof. We start by an explicit description of the polyhedron $\Sigma_\circ$ and the structural maps $\eta_\circ$ on these polyhedra by means of coordinates. Then we construct the spaces $R\Sigma$, equipped with maps $pr : R\Sigma \to \Sigma$ and $\phi : R\Sigma \to K(\mathbf{I}_a,1)$, which satisfy required boundary conditions (32).

The complex stratification of polyhedra J, Σ, $\Sigma_\circ$ by means of the coordinate system angle - momentum

Let us order lens spaces, which form the join, by the integers from 1 up to r and let us denote by $J(k_1,\dots,k_s) \subset J$ the subjoin, formed by a selected set of circles (one-dimensional lens spaces) $S^1/\mathbf{i}$ with indexes $1 \le k_1 < \dots < k_s \le r$, $0 \ge s \ge r$. The stratification above is induced from the standard stratification of the open faces of the standard r-dimensional simplex δ^r under the natural projection $J \to \delta^r$. The preimages of vertexes of a simplex are the lens spaces $J(j) \subset J$, $J(j) \approx S^1/\mathbf{i}$, $1 \le j \le r$, generating the join.

Define the space $J^{[s]}$ as a subspace of J, obtained by the union of all subspaces $J(k_1,\dots,k_s) \subset J$.

Denote the maximum open cell of the space $\hat{p}^{-1}(J(k_1,\dots,k_s))$ by $\hat{U}(k_1,\dots,k_s) \subset S^{n-k}/\mathbf{i}$. This open cell is called an elementary stratum of the depth $(r-s)$. A point at an elementary stratum $U(k_1,\dots,k_s) \subset S^{n-k}/\mathbf{i}$ is defined by a set of coordinates $(\check{x}_{k_1},\dots,\check{x}_{k_s},\lambda)$, where $\check{x}_{k_i} \in S^1$ is a coordinate on the 1-sphere (circle), covering lens space with the number k_i, $\lambda = (l_{k_1},\dots,l_{k_s})$ is a barycentric coordinate on the corresponding $(s-1)$-dimensional simplex of the join. Thus if the two sets of coordinates are identified under the transformation of the cyclic $\mathbf{I}_a$-covering by means of the generator, which is common to the entire set of coordinates, then these sets define the same point on $S^{n-k}/\mathbf{i}$. Points on elementary stratum $\hat{U}(k_1,\dots,k_s)$ belong in the union of simplexes with vertexes belong to the lens spaces of the join with corresponding coordinates. Each elementary strata $\hat{U}(k_1,\dots,k_s)$ is a base space of the double covering $U(k_1,\dots,k_s) \to \hat{U}(k_1,\dots,k_s)$, which is induced from the double covering $\mathbb{RP}^{n-k} \to S^{n-k}/\mathbf{i}$ by the inclusion $\hat{U}(k_1,\dots,k_s) \subset S^{n-k}/\mathbf{i}$.

The polyhedron $K_\circ = K \setminus Q_{diag}$ is slitted into the union of open subsets (elementary strata) $K(k_1,\dots,k_s)$, $1 \le s \le r$ correspondingly with the stratification

$$J^{(r)} \subset \dots \subset J^{(1)} \subset J^{(0)}, \tag{37}$$

of the space J. For the considered stratum a number $r-s$ of missed coordinates to the full set of coordinates is called the deep of the stratum.

Let us introduce the following denotation:

$$J^{[i]} = J^{(i)} \setminus J^{(i+1)}. \tag{38}$$

The polyhedron $\Sigma_\circ$ is defined as the union of $K_\circ$ with the antidiagonal stratified subpolyhedron (24) over the common subpolyhedron $Q_{antydiag}$.

Let us describe an elementary stratum $K^{[r-s]}(k_1,\dots,k_s)$ by means of the coordinate system. To simplify the notation let us consider the case $s=r$.

Suppose that for a pair of points $(x_1,\, x_2)$, defining a point on $K^{[0]}(1,\dots,r)$, the following pair of points $(\check{x}_1, \check{x}_2)$ on the covering space S^{n-k} is fixed, and the pair $(\check{x}_1, \check{x}_2)$ is mapped to the pair $(x_1,\, x_2)$ by means of the projection of $S^{n-k} \to \mathbb{RP}^{n-k}$. Accordingly to the construction above, we denote by $(\check{x}_{1,i}, \check{x}_{2,i})$, $i = 1,\dots,r$ a set of spherical coordinates of each point. Each such coordinate with the number i defines a point on 1-dimensional sphere (circle) S^1_i with the same number i, which covers the corresponding circle $J(i) \subset J$ of the join. Note that the pair of coordinates with the common number determines the pair of points in a common layer of the standard cyclic $\mathbf{I}_a$-covering $S^1 \to S^1/\mathbf{i}$.

The collection of coordinates $(\check{x}_{1,i}, \check{x}_{2,i})$ are considered up to independent changes to the antipodal. In addition, the points in the pair (x_1, x_2) does not admit a natural order and the lift of the point in K to a pair of points $(\bar{x}_1, \bar{x}_2)$ on the sphere S^{n-k}, is well determined up to 8 different possibilities. (The order of the group $\mathbf{D}_4$ is equal to 8.)

An analogous construction holds for points on deeper elementary strata $K^{[r-s]}(k_1,\dots,k_s)$, $1 \le s \le r$.

The coordinate description of elementary strata of the polyhedron $K_\circ \subset \Sigma_\circ$

Let $x \in K^{[r-s]}(k_1,\dots,k_s)$ be a point on an elementary stratum. Consider the sets of spherical coordinates $\check{x}_{1,i}$ и $\check{x}_{2,i}$, $k_1 \le i \le k_s$ of the point x. For each i the following cases: a pair of i-th coordinates coincides; antipodal, the second coordinate is obtained from first by the transformation by means of the generator (or by the minus generator) of the cyclic cover. Associate to an ordered pair of coordinates $\check{x}_{1,k_i}$ and $\check{x}_{2,k_i}$, $1 \le i \le s$ the residue v_i of a value $+1$, -1, $+\mathbf{i}$ or $-\mathbf{i}$, respectively.

When the collection of coordinates of a point is changed to the antipodal collection, say, the collection of coordinates of the point x_2 is changed to the antipodal collection, the set of values of residues of the new pair $(\bar{x}_1, \bar{x}_2)$ on the spherical covering is obtained from the initial set of residues by changing of the signs. The residues of the renumbered pair of points change by the inversion. Obviously, the set of residues does not change, if we choose another point on the same elementary stratum of the space $K_\circ$.

Elementary strata of the space $K(k_1,\dots,k_s)$, in accordance with sets of residues, are divided into 3 types: $\mathbf{I}_a, \mathbf{I}_b, \mathbf{I}_d$. If among the set of residues are only residues $\{+\mathbf{i}, -\mathbf{i}\}$ (respectively, only residues $\{+1,-1\}$), we shall speak about the elementary stratum of the type $\mathbf{I}_a$ (respectively of the type $\mathbf{I}_b$). If among the residues are residues from the both set $\{+\mathbf{i}, -\mathbf{i}\}$ and $\{+1,-1\}$, we shall speak about elementary stratum of the type $\mathbf{I}_d$. It is easy to verify that the restriction of the structure mapping $\eta : K_{0\circ} \to K(\mathbf{D}_4, 1)$ on an elementary stratum of the type $\mathbf{I}_a, \mathbf{I}_b, \mathbf{I}_d$ is represented by the composition of a map in the space $K(\mathbf{I}_a, 1)$ (respectively in the space $K(\mathbf{I}_b, 1)$ or $K(\mathbf{I}_d, 1)$) with the map $i_a : K(\mathbf{I}_a, 1) \to K(\mathbf{D}_4, 1)$ (respectively, with the map $i_b : K(\mathbf{I}_b, 1) \to K(\mathbf{D}_4, 1)$ or $i_d : K(\mathbf{I}_d, 1) \to K(\mathbf{D}_4, 1)$). For the first two types of strata the reduction of the structural mapping (up to homotopy) is not well defined, but is defined only up to a composition with the conjugation in the subgroups $\mathbf{I}_a$, $\mathbf{I}_b$.

The polyhedron $\Sigma_\circ$ is defined by the union of $K_\circ$ with all antidiagonal strata. On antidiagonal strata the residue of each coordinate is equal to $+\mathbf{i}$. Antidiagonal strata will be considered as strata of the type $\mathbf{I}_a$. The polyhedron Σ is defined from $\Sigma_\circ$ by completion by all elementary diagonal strata (on each elementary stratum $K(k_1, \dots, k_s)$ the residue of each coordinate is equal to $+1$), of the boundary of the polyhedron. It is easy to check that $\Sigma \setminus \Sigma_\circ$ contains all elementary diagonal strata of deeps ≥ 1.

Define the following open subpolyhedra:

$$K_{\mathbf{I}_a\circ} \subset K_\circ \subset \Sigma_\circ, \tag{39}$$

$$K_{\mathbf{I}_b\circ} \subset K_\circ \subset \Sigma_\circ, \tag{40}$$

$$K_{\mathbf{I}_d\circ} \subset K_\circ \subset \Sigma_\circ \tag{41}$$

by the union of all elementary strata of the corresponding type.

Description of the structural map $\eta_\circ : \Sigma_\circ \to K(\mathbf{D}, 1)$, by means of the coordinate system

Let $x = [(x_1, x_2)]$ be a marked a point on $K_\circ$, on a maximal elementary stratum. Consider closed path $\lambda : S^1 \to K_\circ$, with the initial and ending points in this marked point, intersecting the singular strata of the depth 1 in a general position in a finite set of points. Let $(\check{x}_1, \check{x}_2)$ be the two spherical preimages of the point x. Define another pair $(\check{x}'_1, \check{x}'_2)$ of spherical preimages of x, which will be called coordinates, obtained in result of the natural transformation of the coordinates $(\check{x}_1, \check{x}_2)$ along the path λ.

At regular points of the path λ the family of pairs of spherical preimages in the one-parameter family is changing continuously, that uniquely identifies the inverse images of the end point of the path by the initial data. When crossing the path with the strata of depth 1, the corresponding pair of spherical coordinates with the number l is discontinuous. Since all the other coordinates remain regular, the extension of regular coordinates along the path at a critical moment time is uniquely determined. For a given point x on elementary stratum of the depth 0 of the spaces $K_\circ$ the choice of at least one pair of spherical coordinates is uniquely determines the choice of spherical coordinates with the rest numbers. Consequently, the continuation of the spherical coordinates along a path is uniquely defined in a neighborhood of a singular point of the path.

The transformation of the ordered pair $(\check{x}_1, \check{x}_2)$ to the ordered pair $(\check{x}'_1, \check{x}'_2)$ defines an element the group $\mathbf{D}$. This element does not depend on the choice of the path l in the class of equivalent paths, modulo homotopy relation in the group $\pi_1(\Sigma_\circ, x)$. Thus, the homomorphism $\pi_1(\Sigma_\circ, x) \to \mathbf{D}$ is well defined and the induced map

$$\eta_\circ : \Sigma_\circ \to K(\mathbf{D}, 1) \tag{42}$$

coincides with structural mapping, which was determined earlier. It is easy to verify that the restriction of the structural mapping η on the connected components of a single elementary stratum $K_\circ(1, \dots, r)$ is homotopic to a map with the image in the subspeces $K(\mathbf{I}_a, 1)$, $K(\mathbf{I}_b, 1)$, $K(\mathbf{I}_d, 1)$, which corresponds to the type and subtype elementary stratum.

Consider an elementary stratum $K^{[r-s]}(k_1, \dots, k_s) \subset K_\circ^{(r-s)}$ of the depth of $(r-s)$. Denote by

$$\pi : K^{[r-s]}(k_1, \dots, k_s) \to K(\mathbb{Z}/2, 1) \tag{43}$$

the classifying map, that is responsible for the permutation of a pair of points around a closed path on this elementary stratum. This mapping is called the classifying mapping for the canonical 2-sheeted covering.

The mapping π coincides with the composition

$$K^{[r-s]}(k_1, \dots, k_s) \stackrel{\eta}{\longrightarrow} K(\mathbf{D}_4, 1) \stackrel{p}{\longrightarrow} K(\mathbb{Z}/2, 1),$$

where $K(\mathbf{D}_4, 1) \stackrel{p}{\longrightarrow} K(\mathbb{Z}/2, 1)$ be the map of the classifying spaces, which is induced by the epimorphism $\mathbf{D}_4 \to \mathbb{Z}/2$ with kernel $\mathbf{I}_c \subset \mathbf{D}_4$. The canonical 2-sheeted covering, which is associated with the mapping π, denote by

$$\bar{K}^{[r-s]}(k_1, \dots, k_s) \to K^{[r-s]}(k_1, \dots, k_s). \tag{44}$$

Lemma 5. *The restriction of the mapping* (43) *on each elementary stratum is homotpic to the composition*

$$\pi : K^{[r-s]}(k_1, \dots, k_s) \to S^1 \subset K(\mathbb{Z}/2, 1), \tag{45}$$

where $S^1 \subset K(\mathbb{Z}/2, 1)$ is the embedding of 1-dimensional skeleton of the classifying space.

Proof of Lemma 5

Explicit formula for the mapping (45) is required. Let us considered cases of elementary strata of types $\mathbf{I}_b$, $\mathbf{I}_a$, $\mathbf{I}_d$.

An arbitrary point $(x_1, x_2) \in K^{[r-s]}(k_1, \dots, k_s)$ is uniquely determined by an equivalent class of the collection of the angular and momentum coordinates. Two *prescribed* pair of angular coordinates beside all pairs of angular coordinates are described below. Let us define the mapping (43) by two pairs of angular coordinates, the first pair of coordinates has the residue $+1$, the second pair has the residue -1.

Correspondingly values of the residues, we shall denote the first pair of prescribed angular coordinates by $(\check{x}_{1,-}, \check{x}_{2,-})$, the second pair of prescribed coordinates by $(\check{x}_{1,+}, \check{x}_{2,+})$. Each prescribed coordinate $\check{x}_{1,-}$, $\check{x}_{1,+}$, $\check{x}_{2,-}$, $\check{x}_{2,+}$ determines the corresponding point on S^1. It is easy to check that $\check{x}_{1,-} = -\check{x}_{2,-}$, $\check{x}_{1,+} = \check{x}_{2,+}$. Therefore the mappings $[(x_1, x_2)] \mapsto \check{x}_{1,-} \pmod{-1}$, $[(x_1, x_2)] \mapsto \check{x}_{1,+} \pmod{-1}$ are well defined and, in particular, are not dependent of an order of the points in the pair.

Denote the product $\check{x}_{1,-} \cdot \check{x}_{1,+}$ of two points on S^1 by $\check{x}$. The mapping $(x_1, x_2) \mapsto \check{x} \pmod{-1} \in S^1/-1$ is well defined. This mapping is the restriction of the required mapping (45) on the stratum of type $\mathbf{I}_b$.

Assume that a point $(x_1, x_2) \in K^{[r-s]}(k_1, \dots, k_s)$ belongs to the stratum of the type $\mathbf{I}_a$ (including antidiagonal strata). The mapping (43) is uniquely determined by a transformation of the two prescribed pairs of angular coordinates with residues $-\mathbf{i}$, $+\mathbf{i}$.

Residues of a pair of the prescribed coordinates are well defined, let us denote the first pair of the prescribed coordinates by $(\check{x}_{1,-\mathbf{i}}, \check{x}_{2,-\mathbf{i}})$ and the second pair of the prescribed coordinates by $(\check{x}_{1,+\mathbf{i}}, \check{x}_{2,+\mathbf{i}})$. It is not hard to check that $\mathbf{i}\check{x}_{1,-} = \check{x}_{2,-}$, $\mathbf{i}\check{x}_{1,+} = \check{x}_{2,+}$. The mapping $[(x_1, x_2)] \mapsto (\check{x}_{1,+\mathbf{i}})^2 \pmod{-1}$ is well defined. This mapping is the restriction of the required mapping (45) on the stratum of type $\mathbf{I}_a$.

Assume that a point $(x_1, x_2) \in K^{[r-s]}(k_1, \dots, k_s)$ belongs to the stratum of the type $\mathbf{I}_d$ The mapping (43) is homotopic to the constant. Let us define this mapping by the following formula. Take a coordinate system such that there exist the prescribed pair of angular coordinates with the residue $+\mathbf{i}$. Denote this prescribed pair of coordinates by $(\check{x}_{1,+\mathbf{i}}, \check{x}_{2,+\mathbf{i}})$. The mapping $[(x_1, x_2)] \mapsto (\check{x}_{1,+\mathbf{i}})^2 \pmod{-1}$ is the restriction of the required mapping (45) on the stratum of type $\mathbf{I}_d$.

Lemma 5 is proved.

Free involution of the polyhedron $K_\circ$

Let us consider the polyhedron $K_\circ$, which is defined by the formula (25). Define a free involution

$$T_{K_\circ} : K_\circ \to K_\circ, \tag{46}$$

which transforms neighborhoods of the diagonal and of the antidiagonal it they selfs. Define the involution (46) by the formula $T_{K_\circ}(\check{x}_1, \check{x}_2, \lambda) = (\mathbf{i}\check{x}_1, \mathbf{i}\check{x}_2, \lambda)$.

Real (double) stratification of the polyhedron J

Let us consider the polyhedron J (the standard sphere) and the stratification of this polyhedron, which is defined by the formula (37). For each elementary stratum of this stratification let us define an additional stratification. This (double) stratification is called the real stratification. Let us re-denote $J^{(r-s)}(k_1, \dots k_s)$ by $J^{(r-s,0)}(k_1, \dots k_s)$ and consider on $J^{(r-s,0)}(k_1, \dots k_s)$ the "'angle-momentum"' coordinate system. Let us define the following stratification

$$J^{(r-s,s)}(k_1, \dots k_s) \subset J^{(r-s,s-1)}(k_1, \dots k_s) \subset \dots \subset J^{(r-s,0)}(k_1, \dots k_s), \tag{47}$$

which consists of a family of embedding polyhedra of codimension 1. Denote by $J^{(r-s,i)}(k_1, \dots k_s)$ the subpolyhedron in $J^{(r-s,0)}(k_1, \dots k_s)$ of all points, such that not less then i points are equal to $+1$. Denote the polyhedron $J^{(r-s,i)}(k_1, \dots k_s) \setminus J^{(r-s,i+1)}(k_1, \dots k_s)$ by $J^{[r-s,i]}(k_1, \dots k_s)$. It is not difficult to check that the polyhedron $J^{[r-s,i]}(k_1, \dots k_s)$ is the disjoin union of connected strata of the stratification (47), for each connected strata of $J^{[r-s,i]}(k_1, \dots k_s)$ the collection of i (singular) angular coordinates is fixed, each coordinate from this collection is equal to $+1$ or to -1, the last (regular) coordinates could be arbitrary in $S^1 \setminus \{+1, -1\}$. Angular singular coordinates are called *auxiliary*, regular angular coordinates are called *principal.* In particular, for points in $J^{[r-s,r-s]}(k_1, \dots k_s)$ all angular coordinates are auxiliary.

Real (double) stratification of the polyhedron $K_\circ$

Recall that an elementary stratum

$$K^{[r-s]}(k_1, \dots, k_s) \tag{48}$$

is defined as the inverse image of the elementary stratum $J^{[r-s]}(k_1, \dots, k_s)$ by the natural projection of a singularity on its image. Let us define the following stratification

$$K^{[r-s,i]}(k_1,\dots,k_s) \subset K_\circ^{[r-s]} \subset K_\circ \qquad (49)$$

of an arbitrary elementary stratum of the type $\mathbf{I}_b$ as the stratification, which is induced from the stratification (47) by this natural projection.

For an arbitrary connected stratum of the polyhedron $K^{[r-s,i]}(k_1,\dots k_s)$ a collection of i pairs of (singular) angular coordinates are fixed. Each coordinate in a pair of the collection takes one of the following two pairs of values $\{+1,-1\}$, $\{+\mathbf{i},-\mathbf{i}\}$; this singular coordinates are called auxiliary. Hence, the auxiliary coordinates are divided into real and imaginary. The last (regular) coordinates are called principal.

An pair of principle coordinates takes the value from one of the two intervals $(0,\frac{\pi}{2}) \cup (\pi,\frac{3\pi}{2})$, or from one of the two intervals $(\frac{\pi}{2},\pi) \cup (\frac{3\pi}{2},2\pi)$.

The involution (46) changes each imaginary (real) coordinate into real (imaginary), this involution acts free on the set of elementary strata of the stratification (49). Strata are divided into pairs, the involution (46) changes the two strata in a pairs.

Prescribed coordinate system of a real stratification (49) **of the polyhedron** $K_{\mathbf{I}_b\circ}$

Let us recall that the space $K_{\mathbf{I}_b\circ}$ is decomposed into the union of closures $Cl(K^{[r-s,i]}(k_1,\dots,k_s))$, $0 \le s \le r$ of strata of the real stratification (49) (the closures are considered into the space $K_\circ$). Recall that residues of coordinates take values into $\{+1,-1\}$.

On each elementary stratum of the stratification (49) let us define a prescribed coordinate system. This prescribed coordinate system is defined in a marked point $x = ([x_1,x_2])$ by the ordered pair of the spherical preimages $(\check{x}_1,\check{x}_2)$ up to the transformation $(\check{x}_1,\check{x}_2) \mapsto (-\check{x}_1,-\check{x}_2)$. In particular, the equivalent class of a prescribed coordinate system is well defined with respect to transformations $(\check{x}_1,\check{x}_2) \mapsto (\check{x}_2,-\check{x}_1)$, this transformation is given by the action of the generator of the subgroup $\mathbf{I}_a \subset \mathbf{D}$ on the coordinates of the point $x \in K_{\mathbf{I}_b\circ}$. The involution (46) transforms the prescribed coordinate system on an elementary stratum into a coordinate system on the image of this elementary stratum. The prescribed coordinates has to be invariant with respect to this transformation.

The prescribed coordinate system on the stratum of the stratification (48) **of the polyhedron** $K_{\mathbf{I}_a\circ} \cup \Sigma_{antidiag}$

Let us recall that the space $K_{\mathbf{I}_a\circ} \cup \Sigma_{antidiag}$ is decomposed into the union of closures $Cl(K^{[r-s]}(k_1,\dots,k_s))$, $0 \le s \le r$ of strata of the stratification (48) (the closures are considered into the space $K_\circ$; antidiagonal strata are also possible). Recall that residues of coordinates take values into $\{+\mathbf{i}, -\mathbf{i}\}$.

On each elementary stratum let us define a prescribed coordinate system such that the number of residues, which is equal to $-\mathbf{i}$, is not less then half number of residues. In the case when the half of residues are equal to $+\mathbf{i}$ and the last half of residues are equal to $-\mathbf{i}$, the prescribed coordinate system is fixed such that the value of the residue of the coordinate of the smallest number is equal to $+\mathbf{i}$. This rule is considered for antidiagonal strata.

The involution (46) transforms the prescribed coordinate system on an elementary stratum into a coordinate system on the image of this elementary stratum (recall that on the anti diagonal stratum this transformation is fixed). The prescribed coordinates has to be invariant with respect to this transformation. A prescribed coordinate system is uniquely defined up to the transformation $(\check{x}_1, \check{x}_2) \mapsto (\check{x}_2, -\check{x}_1)$. The prescribed coordinate system on a stratum of (49) is defined by the restriction of the prescribed coordinate system of (48).

The prescribed coordinate system on the stratum of the stratification (48) of the polyhedron $K_{\mathbf{I}_d\circ}$

Let us recall that the space $K_{\mathbf{I}_d\circ}$ is decomposed into the union of closures $Cl(K^{[r-s]}(k_1, \dots, k_s))$, $0 \le s \le r$ of strata of the stratification (48) (the closures are considered into the space $K_\circ$). Recall that residues of coordinates take values into $\{+\mathbf{i}, -\mathbf{i}, +1, -1\}$.

On each elementary stratum let us define a prescribed coordinate system such that the number of residues, which is equal to $-\mathbf{i}$, is not less then half number of residues, which take the values in $\{+\mathbf{i}, -\mathbf{i}\}$. In the case when the half of the considered residues are equal to $+\mathbf{i}$ and the last half of residues are equal to $-\mathbf{i}$, the prescribed coordinate system is fixed such that the value of the residue of the coordinate of the smallest number is equal to $+\mathbf{i}$. A prescribed coordinate system is uniquely defined up to the transformation $(\check{x}_1, \check{x}_2) \mapsto (\check{x}_2, -\check{x}_1)$. The prescribed coordinate system on a stratum of (49) is defined by the restriction of the prescribed coordinate system of (48).

Allowable pairs of strata

For an arbitrary elementary stratum $\beta \subset K^{[r-s,i]}(k_1, \dots, k_s)$ of the polyhedron $K_\circ$ let us consider all smallest elementary strata, which belong to the boundary of the closure $Cl(K^{[r-s,i]}(k_1, \dots, k_s))$.

Let us consider an elementary stratum $\alpha \subset Cl(\alpha) \subset Cl(K^{[r-s,i]}(k_1,\dots,k_s))$, $\alpha \ne \beta$. Write $\alpha \prec \beta$ for short. Let us restrict the prescribed coordinate system of β to the stratum α. Assuming the restricted coordinate system of β on α is equivalent to the prescribed coordinate system on α itself, we shall call that the pair $\alpha \prec \beta$ is allowable. Oppositely, if the restricted coordinate system of β on α is equivalent to the prescribed coordinate system on α itself, we shall call that the pair $\alpha \prec \beta$ is not allowable.

Let us consider more details. The pair $\alpha \prec \beta$ is allowable, if the following condition is satisfied. Let $a = (a_1, a_2) \in \alpha$, $b = (b_1, b_2) \in \beta$; $(\check{a}_1, \check{a}_2)$, $(\check{b}_1, \check{b}_2)$ are the prescribed coordinate systems at the corresponding points. This two systems are related by means of a one of the following transformations:

$$(\check{b}_1, \check{b}_2) \mapsto \pm(\check{b}_1, \check{b}_2) = (\check{a}_1, \check{a}_2), \tag{50}$$

$$(\check{b}_1, \check{b}_2) \mapsto \pm(\check{b}_2, \check{b}_1) = (\check{a}_1, \check{a}_2), \tag{51}$$

$$(\check{b}_1, \check{b}_2) \mapsto \pm(-\check{b}_1, \check{b}_2) = (\check{a}_1, \check{a}_2), \tag{52}$$

$$(\check{b}_1, \check{b}_2) \mapsto \pm(\check{b}_2, -\check{b}_1) = (\check{a}_1, \check{a}_2). \tag{53}$$

Let us assume that β is a stratum of the type $\mathbf{I}_b$. Then α is a stratum of the same type. Assume that the both strata are inside a common elementary stratum of the stratification (48). In this case transformations (50), (51) are possible. Transformations of the type (50) preserve the prescribed coordinate systems, in this case the considered pair of the strata is allowable. Transformation of the type (51) change the prescribed coordinate systems, in this case the considered pair of the strata is not allowable.

Let us assume that β is a stratum of the type $\mathbf{I}_a$. Then α is a stratum of the same type. The considered pair of strata is a pair of neighbor strata of the stratification (48). Transformations of all types are possible. Transformations of the types (50), (53) preserve the prescribed coordinate systems, in this case the pair of the considered strata is allowable. Transformations of the type (51), (52) change the prescribed coordinate systems, in this case the pair of the considered strata is not allowable.

Let us assume that β is a stratum of the type $\mathbf{I}_d$. Then α is a stratum of any possible type. Assume that α is the stratum of the type $\mathbf{I}_a$. Then the considered pair of strata is a pair of neighbor strata of the stratification (48). Then the pair $\alpha \prec \beta$ is allowable and the transformation of the type (50) is only possible. Assume that α is the stratum of the type $\mathbf{I}_b$, or

$\mathbf{I}_d$. Then transformations of an arbitrary type are possible. In the case α is of the type $\mathbf{I}_d$, the considered pair of strata is a pair of neighbor strata of the stratification (48). Transformations of the types (50), (53) preserve the prescribed coordinate systems, in this case the pair of the considered strata is allowable. Transformations of the type (51), (52) change the prescribed coordinate systems, in this case the pair of the considered strata is not allowable.

The space $Y_\circ$

For an arbitrary pair $\alpha \prec \beta$, let us define an open conical ε-neighborhood of the stratum $\beta \subset K^{[r-s,i]}(k_1,\dots,k_s) \subset \Sigma_\circ$, which we will denote by $C(\alpha,\beta;\varepsilon) \subset \alpha$.

The cone of the (smallest) strata α inside β is defined as an open domain, which is the open cone (of a small height ε) over the interiors of the closure of the union of all smallest open cones, which belong to $Cl(\beta)$. Let us denote by $Con'(\alpha,\beta;\varepsilon)$ the cone of the stratum α in the stratum $Cl(\beta)$.

In the case $\alpha \prec \beta$ of the codimension 2 by means of the stratification (48), or in the case of the codimension 1 by means of the stratification (49), we shall call the cone $Con'(\alpha,\beta;\varepsilon)$ is elementary. An elementary cone of (48) is defined by means of the cone inside the standard simplex of the momentum coordinates. An elementary cone of (49) is defined by means of the cone inside the standard $\mathbb{R}^r$, determined locally angle coordinates. In a general case a non-elementary cone is defined by the corresponding sequence of elementary cones.

For an arbitrary elementary cone $Con'(\alpha,\beta;\varepsilon)$ let us define the corresponding elementary ε–cone. Let us denote this elementary ε–cone by $Con(\alpha,\beta;\varepsilon)$. The coordinates on $Con(\alpha,\beta;\varepsilon)$ are divided into degenerate and non-degenerate. The non-degenerated coordinates coincide with non-degenerate coordinates of the cone $Con'(\alpha,\beta;\varepsilon)$. The degenerate coordinates are all the last angle-momentum coordinates, such that an arbitrary degenerate coordinate is ε–small. This means that a degenerate angle coordinate takes a value in the interval $(0,\varepsilon)$, (or in the interval $(-\varepsilon,0)$) up to an additive constant $\pm\frac{\pi}{2}$, or $+\pi$. A degenerate momentum coordinate takes a value in the standard ε–neighborhood of the corresponding face (this face is associated with the momentum coordinates on β) of the standard r-simplex of the all momentum coordinates. By the construction we get $Con'(\alpha,\beta;\varepsilon) \subset Con(\alpha,\beta;\varepsilon)$.

For an arbitrary pair of strata $\alpha \prec \beta_0$ let us define the conical ε–neighborhood

$$C(\alpha,\beta_0;\varepsilon) = Cl(\cup_{j,j\neq 0} Con(\alpha,\beta_j;\varepsilon)), \tag{54}$$

as the union of closures of all elementary strata $\alpha \prec \beta_j \prec \beta_0$, $\beta_j \neq \beta_0$. Evidently, if $\alpha \prec \beta_1 \prec \beta_0$, then $C(\alpha,\beta_0;\varepsilon) \subset C(\alpha,\beta_1;\varepsilon)$, because in the stratum β_0 one has additional restrictions for degenerate coordinates of the stratum β_1, which are non-degenerate on the stratum β_0.

Let us define the polyhedron $Y_\circ(\varepsilon)$ as the result of a removal from $\Sigma_\circ$ of conical ε–neighborhoods of all non-allowable pairs of strata. By construction $Y_\circ(\varepsilon)$ is obtain from $\Sigma_\circ$ by the removal of an open polyhedron. Therefore the polyhedron $Y_\circ(\varepsilon)$ is closed into $\Sigma_\circ$.

The value ε in the presented construction is not important, because the homotopy type of the space $Y_\circ(\varepsilon)$ is not dependent on ε, if ε is small enough. Let us modify the construction, such that the modified construction is independent on ε.

Let us consider the polyhedron $\Sigma_\circ \subset \Gamma_\circ$ and let us define by $X(\varepsilon,\varepsilon_1)_\circ$ a small regular ε_1–neighborhood of the subpolyhedron $Y_\circ(\varepsilon) \subset \Sigma_\circ \subset \Gamma_\circ$.

Evidently, the inclusion

$$Y_\circ(\varepsilon) \subset X_\circ(\varepsilon,\varepsilon_1). \tag{55}$$

is well defined and for $0 < \varepsilon_1 << \varepsilon << 1$ this inclusion is a homotopy equivalence.

Let us define the following space

$$Y_\circ = \lim_{\varepsilon\to 0,\varepsilon_1\to 0} X_\circ(\varepsilon,\varepsilon_1), \tag{56}$$

where the direct limit is taken for $0 < \varepsilon << \varepsilon_1$.

Definition of the subspace $R\Sigma$ in Lemma 3

Define the space $R\Sigma$ as a subspace in Σ, which is defined as the result of the union of the space $Y_\circ$ (see the formula (56)) with the subspace $Q_{diag} \subset \Sigma$.

The boundary subspace RQ_{diag} of the resolution space $R\Sigma$ and the subspaces $RN \subset R\Sigma$, $RN \subset R\Sigma$

Let us define the boundary subspace $RQ_{diag} \subset R\Sigma$, which is used in the diagram (32). Let us define $RQ_{diag} = (Q_{diag} \cap R\Sigma) \subset R\Sigma$. Let us define $R\Sigma_\circ = R\Sigma \setminus Q_{diag}$. Let us define the subspaces in the formula (28) by the following formula: $RN = R\Sigma$, $RL = U(RQ_{diag})$.

Resolution mapping $\phi_\circ : R\Sigma_\circ \to K(\mathbf{I}_a, 1)$, proof of Lemma 3

Let us restrict the structural mapping $\eta_\circ$, which is given by the formula (23), on the subspaces $Y_{\mathbf{I}_b} \cap R\Sigma$, $Y_{\mathbf{I}_a} \cap R\Sigma$, $Y_{\mathbf{I}_d} \cap R\Sigma$, $\Sigma_{antidiag} \cap R\Sigma$. All this restrictions are homotopic to a mapping into the subspace $K(\mathbf{I}_a, 1) \subset K(\mathbf{D}, 1)$. The mapping $\kappa : U(RQ_{diag}) \to K(\mathbf{I}_d, 1) \subset K(\mathbf{I}_a, 1)$ is well defined The required mapping $\phi : R\Sigma_\circ \to K(\mathbf{I}_a, 1)$ is defined as the result of the gluing of the considered mappings on the subspaces. Evidently, the boundary conditions, given by the diagram (32) is satisfied. Lemma 3 is proved.

The last step of the proof of Lemma 4; the deformation $i_2 \circ c_1' \mapsto d$

Let us consider the sphere $J \subset \mathbb{R}^n$ inside the hyperspace of the codimension 4: $J \subset \mathbb{R}^{n-4} \subset \mathbb{R}^n$. Let us consider the cylinder $J \times I \subset \mathbb{R}^{n-4} \times I \subset \mathbb{R}^n \times I$.

Let us denote the standard projection $\mathbb{R}^n \to \mathbb{R}^{n-4}$ by $proj$. Let us decomposed the space $\mathbb{R}^n$ as the Cartesian product: $\mathbb{C}^2 \times \mathbb{R}^{n-4} \cong \mathbb{R}^n$, in particular, the fiber of the projection $proj$ is equipped with the complex structure.

The standard projection of the regular neighborhood $U(J)$ of the sphere $J \subset \mathbb{R}^n$ on the central sphere $i_J : J \subset \mathbb{R}^n$ was denoted above by $proj_J$. Let us decomposed the projection $proj_J : U(J) \to J$ into the following composition: $proj_J = proj_1 \circ proj|_{U(J)}$, where the projection $proj$ (on the codimension 4) was defined above, the projection $proj_1 : \mathbb{R}^{n-4} \cap U(J) \to J$ is the standard projection (of the codimension $n-k-4$).

Recall that the deformation $i_1 \circ c \mapsto c'$ was constructed as a $proj_J$–vertical generic projection inside the subspace $\mathbb{R}^{n-4} \subset \mathbb{R}^n$. In the denotations introduced above this deformation is a $proj_1$–vertical generic deformation.

Lemma 6. *Assuming the dimensional restriction* (2)*, there exists an arbitrary C^0–small deformation $\tau : c' \mapsto d$ (the mapping d is the required mapping in Lemma* 2*,A), such that the following condition (C) is satisfied:*

(C) for a suitable values $0 < \varepsilon_1 << \varepsilon << 1$ the polyhedron $N_\circ$ of self-intersection points of the mapping d is contained inside a small regular neighborhood $X_\circ(\varepsilon, \varepsilon_1)$ of the polyhedron $Y_\circ$, which was defined by the formula (8).

Proof of Lemma 6[1]

The stratification (48) of the polyhedron $\Sigma_\circ$, which is agree with (37) by the projection $proj_1$ is well defined. Additionally, the stratification of the subpolyhedron $K_{\mathbf{I}_b\circ} \subset \Sigma_\circ$ is well defined. Denote by $T(\Sigma_\circ^{[s]})$ the tangent

[1]this proof was discussed at the topological seminar of Prof. S.A.Bogaty

space to the corresponding stratum of the deep s of the stratification (38), inside which the considered stratum $\Sigma_{\circ}^{[s]}$ is projected by $proj_1$.

In each point $y \in \Sigma_{\circ}^{[s]}$ define an orthogonal to $T(\Sigma_{\circ}^{[s]})$ complex vector space $R(y)^{[s]}$ of real dimension $2(s+2)$. The space $R(y)^{[s]}$ define as the direct sum $R_1(y) \oplus R_2^{[s]}(y)$, where $R_1(y)$ is the vector space of real dimension 4, parallel to the subspace [2] in the fiber of the projection $proj$, $R_2^{[s]}(y)$ is the complex vector space of real dimension $2s$ which is parallel to the tangent space $T(J)$ of the sphere (this tangent space is embedded into the image of the projection $proj$) and orthogonal to $T(\Sigma_{\circ}^{[s]})$. For an arbitrary point $y \in \Sigma_{\circ}^{[s]}$ the complex vector spaces $R_1(y)$, $R_2^{[s]}(y)$ are uniquely defined.

The standard base of the complex vector space $R_2^{[s]}(y)$, associated with the spaces, which are orthogonal to the strata (37) of J (the number of a basis vector corresponds to the deep of the stratum). At a point $y \in \Sigma_{\circ}^{[s]}$ let us define a complex linear subspace $E(y)^{[s]} \subset R(y)_2^{[s]}$. The family of complex linear subspaces $E(y)^{[s]}$, which are indexed by points y have to satisfied the following condition.

The space $E(y)^{[s]}$ in defined by the common linear combination of the vectors of the standard basis of the space $R(y)^{[s]}$, i.e. independently of a choice of the point y at the given stratum α of the deep s of the stratification (48). (The construction of the family $\{E(y)^{[s]}\}$ is given by an induction over the deep s. Therefore instead of $E(y)^{[s]}$ we shall write $E^{[s]}(\alpha)$ of $E(\alpha)$.

Let us define the space $E(\alpha)^{[s]}$, assuming, that the family of the spaces on strata of the less deep is defined. Let y be a point on the stratum α of the deep s, and assume that this point is outside of deepest strata of the stratification (48). For the point y let us consider a sequence of s points

$$\{x_i \in \beta_i\}, \quad 0 \le i \le s-1, \tag{57}$$

which satisfy the following relation:

$$\alpha \prec \beta_{s-1} \prec \cdots \prec \beta_{i+1} \prec \beta_i \prec \cdots \prec \beta_0.$$

In this formula the stratum α is replaced by β_s, if the case $i = s$ is also possible.

Let us define a parameter *ind*, which depends of a sequence (57) and for the given sequence takes s different values, which is parametrized by a positive integer parameter j, $0 < j \le s$. The space $E(\alpha)^{[s]}$ is transversal to each space of the following collection of complex subspaces $\{F(\alpha; ind)$ of the space $R^{[s]}$, the real dimension of the corresponding space satisfy the equation $\dim(F(\alpha; ind)) = 2(s+1)$ (in particular, $F(\alpha; ind) \subset R^{[s]}$ is a complex vector subspace of the codimension 1):

$$F(\alpha; ind) = \oplus_{i=0}^{j-1} E(\beta_i)^{[i]} \oplus T(\alpha, \beta_{j-1}), \quad j = s, \dots, 1. \tag{58}$$

In this formula the complex vector space $T(\alpha, \beta_{j-1})$ of the real dimension $2(s-j+1)$ is uniquely defined for $0 < j \le s$ as a complex vector subspace at the point y is in the orthogonal complement to α inside the tangent space of β_{j-1}. For $j = 1$ the complex line $E(\beta_0)^{[0]}$ is a subspace inside $R_1(\beta_0)$, this line space will be denoted below by $E^{[0]}$, because this is the common space for strata of the deep 0.

The proof of the existence of the collection of the linear vector spaces satisfied the prescribed conditions can be proved by the dimension arguments. This proof is omitted.

Evidently, strata of the double stratification (49) is partial ordered with respect to the relation $\prec$. Therefore we may say about the deep of a stratum of the stratifications (49) and (48) of the polyhedron of self-intersection. The maximal deep of strata of the stratification (49) is $2r$. The maximal deep of strata of the stratification (48) is r.

Let us re-denote f' by f_0. Let us define a sequence of infinitesimal numbers $\delta_0, \dots, \delta_{2r}$ (r is the maximal deep of a strata in $T(J)$), such that $\delta_{i+1} = o(\delta_i)$. Define a sequence of C^0–infinitesimal PL–deformations:

$$f_0 \mapsto f_1 \mapsto \dots \mapsto f_{2r} \tag{59}$$

with a support into the regular neighborhood of the diagonal $U(Q_{diag})$, the C^0–caliber of the deformation $f_{i-1} \mapsto f_i$ is equal to (infinitesimal small) number δ_i. This sequence of infinitesimal small deformations one could considered like a sequence of PL–deformations, each next deformation is so C^0–small, then the properties of the mapping from the previous step of the construction are preserved.

The deformation on the step i, $0 \le i \le 2r$ is defined with a support on strata of (49) of the deep i and is fixed on strata of the deep $i+1$. Let α be an arbitrary stratum of the deep i and let $\alpha \prec \beta_{i-1} \prec \dots \beta_1 \prec \beta_0$ is an arbitrary sequence of strata. The following sequence of inclusions

$$C(\alpha, \beta_{i-1}; \delta_i) \supset \dots \supset C(\alpha, \beta_0; \delta_i)$$

is well defined. The deformation inside the union of elementary cones $C(\alpha, \beta_{i-1}; \delta_i) \cap (Con'(\alpha, \beta_{i-1}; \delta_i) \cup_{j=i-1,\dots,1} Con'(\beta_j, \beta_{j-1}; \delta_i))$ (near α) is along the vector space $E(\alpha) \oplus E(\beta_{i-1}) \oplus \dots E(\beta_1) \oplus E(\beta_0)$. Deformations on pairs of elementary strata β, β', $\beta'' \prec \beta$, $\beta'' \prec \beta'$ are agree on β''.

Finlay, the resulting mapping f_{2r} has only self-intersection points (outside a neighborhood of the diagonal), which are inside a regular neighborhood of the polyhedron $R\Sigma_\circ \subset X_\circ(\varepsilon, \varepsilon_1)$, determined by the formula (8). The parameters ε, ε_1 satisfy the following relation $\varepsilon_1 << \varepsilon << \delta_{2r}$ (see the condition (C) in the statement of the lemma).

At the first step the deformation $f_0 \mapsto f_1$ is a deformation with the support on $\Sigma_\circ^{[0]}$, this deformation is fixed on $\Sigma_\circ^{(1)}$.

This deformation is vertical with respect to the linear complex space $E^{[0]}$ over each point at elementary strata $\alpha \subset K^{[0,0]}(k_1, \dots k_r)$ of the space $\Sigma_\circ^{[0]}$ (below we introduce the only index for strata of (49) as the sum of the indexes).

Let us consider the line bundle $l_\alpha : D(l_\alpha) \to \alpha$ (with the standard segment as the fiber), associated with the canonical 2-sheeted covering over the considered elementary stratum. In Lemma 5 a fiberwise monomorphism $\psi_\alpha : D(l_\alpha) \subset \alpha \times E_\alpha^{[0]}$ (below we say $\mathbb{Z}/2$–section for short) into the trivial line complex S^1–bundle over α. The bundle l_α for strata of all types are classified by the corresponding mapping, which is defined in this lemma). The monomorphism ψ_α determines a C^1–small deformation $f_0 \mapsto f_1$ (a reprojection of self-intersection points along the fiber ψ over the family of elementary strata of the deep 0).

Let us extend by linearity the considered deformation into C^1–infinitesimal deformation over the space $\mathbb{RP}^{n-k}$, which is fixed on self-intersection points on $\Sigma_\circ^{[0]}$.

The a C^1–infinitesimal deformation $f_1 \mapsto f_2$ on $\Sigma_\circ^{(1)}$ is defined as the previous step along the prescribed linear (complex) fiber over each stratum of the deep 1. Define a C^0–infinitesimal crosslinking of mappings f_1 with f_2 and let construct the deformation $f_1 \mapsto f_2$ (the mapping f_2 and the deformation $f_1 \mapsto f_2$ are defined only on $\Sigma_\circ^{(1)}$). Let us consider a stratum α of the deep 1 and a stratum β of the deep 0, such that $\alpha \prec \beta$. This pair of strata could be admissible and non-admissible, correspondingly to the formula (50)-(53) and to various type of neighbor of strata in (49). .

Let us define the deformation $f_1 \mapsto f_2$ near each conical neighborhood $C(\alpha, \beta; \delta_1)$ of the greater stratum β of the deep 0 in the case of non-admissible pair $\alpha \prec \beta$. For a maximal stratum of the deep 0 the conical neighborhood $C(\alpha, \beta; \delta_1)$ coincides with the cone $Con(\alpha, \beta; \delta_1)$. For an admissible pair $\alpha \prec \beta$ the deformation inside the prescribed linear space could be arbitrary. The mapping f_2 outside the union of all cones of admissible pairs, generally speaking, could have self-intersection points inside $X(\varepsilon, \varepsilon_1)$.

Assume that the pair of strata $\alpha \prec \beta$ is non-admissible and this strata are inside the common elementary stratum of (48). Let us consider the trivial complex bundle with the fiber $E^{[0]}$ over the cone $Con(\alpha, \beta; \delta_1)$ and let us define a new trivialization of this bundle. Describe first the new trivialization over the boundary of the cone. On the component of the boundary of the cone inside α, which is contractible, the new trivialization is related with the standard trivialization by the central symmetry. On the component of the boundary of the cone inside β, which is also contractible, the new trivialization is the standard trivialization. Let us fix a trivialization of the bundle over the elementary cone $Con_{\circ}(\alpha, \beta; \delta_1)$ with the prescribed boundary conditions.

Assume that the pair of strata $\alpha \prec \beta$ is non-admissible and this strata are a neighbor strata of (48). Let us consider the trivial complex bundle over $Con(\alpha, \beta)_{\circ}$ with the fiber $E^{[1]}_{\alpha} \oplus E^{[0]}$. Let us consider $\mathbb{Z}/2$–sections, which are associated with the canonical covering over the corresponding components of the boundary of $U(\alpha, \beta)_{\circ}$ is the strata $E^{[1]}_{\alpha}$, $E^{[0]}$, constructed in Lemma 5). The boundary conditions are extended to a $\mathbb{Z}/2$–section of the bundle $E^{[1]}_{\alpha} \oplus E^{[0]}$ over $C'(\alpha, \beta)_{\circ}$ by linearity. A crosslinking of the mapping f_2 with the mapping f_1 by means of a deformation $f_1 \mapsto f_2$ for an arbitrary non-admissible pair of strata α and β is well defined by means of the $\mathbb{Z}/2$–cross sections over the collection of elementary cones $C'(\alpha, \beta)_{\circ}$. The mapping f_1 in a neighborhood $C'(\alpha, \beta)_{\circ}$ is deformed along the vector space $E^{[0]} \oplus E^{[1]}(\alpha)$, or along the line space $E^{[0]}$, as described above.

Assume that the pair of strata α, β are admissible. Define the deformation $f_1 \mapsto f'_1$ in a neighborhood of each elementary cone along the prescribed vector spaces arbitrary. The intersection points outside strata of the deeps not less then 2 belong to $X(\varepsilon, \varepsilon_1)$. The mapping f_2 on the strata of the deep 0 and 1 is constructed.

Let us describe a C^0–infinitesimal deformation of the mapping f_2 into a mapping f_3. Supports of all mappings and deformations are in neighborhoods of pairs of strata of the deeps 0 and 2, and of the deep 1 and 2. By this deformation in each conical neighborhood $(\alpha, \beta; \delta_2)$ for a nonadmissible pair of strata there are no self-intersection points.

The main observation using in this construction is following. Let us assume that a stratum α is in the closure of boundaries of strata β and β' (deeps of the strata β and β' could be arbitrary), then $(\alpha, \beta; \delta_2) \cap (\alpha, \beta'; \delta_2) = \emptyset$ and, therefore, if $\alpha \prec \beta_1 \prec \beta$, $\alpha \prec \beta_1 \prec \beta'$, then $\alpha \prec \beta_1$ is admissible (and self-intersection points of β near α could exist), or the both pairs $\beta_1 \prec \beta$, $\beta_1 \prec \beta'$ are admissible (and self-intersection points of β near β_1 and self-intersection points of β' near β could exist. In the opposite case, if

$(\alpha,\beta;\delta_2)\cap(\alpha,\beta';\delta_2)\neq\emptyset$, then up to the re-indexation of strata the following inclusion $\beta' \prec \beta$ is well defined, in this case the considered pair of strata is admissible and, therefore, self-intersection points on $(\alpha,\beta;\delta_2)\cap(\alpha,\beta';\delta_2)$ could exist.

The following 2 cases of non-admissible pair of strata β_0 of the deep 0 (case 1), or 1 (case 2) and α of the deep 2 are possible. In the case 1 the following 3 subcases are possible: a neighboring of a stratum of the deep 0 in the stratification (49) (the subcase $(1,1)$); a neighboring of a stratum of the deep 2 in the stratifications (49) and the stratification (48) (the subcase $(1,2)$); a neighboring of a stratum of the deep 2 in the stratification (48) (the subcase $(1,3)$). In the case 2 the following 4 subcases are possible: a neighboring of a stratum of the stratification (49) of the deep 1 with a stratum of the stratification (49) of the deep 2 (the subcase $(2,1)$); a neighboring of a stratum of the stratification (49) of the deep 1 with a stratum of the stratification (49) of the deep 2 (the subcase $(2,2)$); a neighboring of a stratum of the stratification (48) of the deep 1 with a stratum of the stratifications (48) and (49) of the deep 2 (the subcase $(2,3)$); a neighboring of a stratum of the stratification (48) of the deep 1 with a stratum of the stratifications (48) of the deep 2 (the subcase $(2,4)$). Proofs in all cases are analogous.

Let us consider the case of non-admissible pair of strata α of the deep 2 and β_0 of the deep 0. Let β_1 be an arbitrary stratum of the deep 1, $\alpha \prec \beta_1 \prec \beta_0$. Let α and β_1 are admissible (in this case f_3 could have self-intersection points on $beta_1$ near α). Then the pair β_1 and β_0 is non-admissible. Let us consider the standard auxiliary deformation $f_2 \mapsto f_2'$ in the neighborhood $C(\alpha,\beta_0;\delta_2)\setminus C(\beta_1,\beta_0;\varepsilon)$ along the fiber $E(\beta)$.

Let us assume that the pair (α,β_1) is non-admissible. Then the pair (β_1,β_0) is admissible. Let us consider an axillary deformation $f_2 \mapsto f_2'$ in the neighborhood $C(\alpha,\beta_0;\delta_2)\setminus U(\beta_1;\varepsilon)$, where $U(\beta_1;\varepsilon)$ is a ε–neighborhood of the stratum β inside $\Sigma_\circ$. The considered auxiliary deformation is well-defined along the fiber $E(\beta)$, by this deformation in the considered neighborhood the mapping f_2' is defined by the standard formula along $E(\beta_0)$. In the both cases the deformation $f_2' \mapsto f_3$ is defined in the considered neighborhood as at the first step of the construction.

Let us consider the case of non-admissible pair of strata α of the deep 2 and β_1 of the deep 1. Let β_0 be an arbitrary stratum of the deep 0, such that the pair $\beta_1 \prec \beta_0$ is non-admissible (and, therefore, the pair $\alpha \prec \beta_0$ is admissible. Let us consider the subcase $(2,4)$. The deformation $f_2 \mapsto f_3$ inside the subspace $E(\alpha)\oplus E(\beta_1)\oplus E(\beta)$ in the neighborhood $C(\alpha,\beta_1;\delta_2)\cap C(\beta_1,\beta;\delta_2)$ is defined by linearity.

Let $y \in \alpha$ is a point on a stratum α of the deep 2. Let us consider the sphere $S(y,\varepsilon)$ with the center at y of the radius ε, this sphere has no points of deepest strata inside. Let us consider the intersection $S(y,\varepsilon) \cap Im(f_1)$. By the construction, the space $S(y,\varepsilon) \cap Im(f_1)$ is inside the subspace $(T(\alpha) \oplus T(\alpha,\beta) \oplus E^{[0]}) \cap S(y,\varepsilon)$, where

$T(\alpha)$ is the tangent space to the stratum α,

$T(\alpha,\beta)$ is the orthogonal complement of the stratum α of the deep 2 in the stratum β of the deep 0, $\alpha \prec \beta$,

$E^{[0]}$ is the prescribed linear space, along this space the deformation $f_0 \mapsto f_1$ on the stratum β was constructed at the previous step.

The spaces $E^{[2]}(\alpha) \cap S(y,\varepsilon)$ and $S(y,\varepsilon) \cap Im(f_1)$ have no intersections.

The constant δ_2 has to be small enough, such that the following property is satisfied. Let us consider the intersection $A = S(y,\varepsilon) \cap Im(f_2(C(\alpha,\beta;\delta_2) \cap C(\beta_1,\beta;\delta_2)))$. The space A is inside a sufficiently small neighborhood (note the metric on the sphere is the standard metric of the unite sphere) the diameter of the considered neighborhood is determined by the constant δ_2 and by the thickness of conical neighborhoods of the stratum of the deep 1 inside the stratum of the deep 0) of the vector space:

$$T(\alpha) \oplus E(\alpha,\beta_1) \oplus E^{[1]}(\beta_1) \oplus E^{[0]} \qquad (60)$$

— $T(\alpha)$ is the tangent space to a stratum of the stratification of J, that contains the stratum α of the deep 2, $z \in \alpha$;

—$T(\alpha,\beta_1)$ is the orthogonal complement of α in β_1;

—$E^{[1]}(\beta_1)$ is the linear space of the prescribed collection of spaces, along this space the deformation $f_1 \mapsto f_2$ on the stratum β_1, $\beta_1 \prec \alpha$, of the deep 1 was constructed on the previous step;

—$E^{[0]}$ is a linear space of the prescribed collection, along this space a deformation on a stratum of the deep 0 near α was constructed on the previous step.

The deformation $f_2 \mapsto f_3$ on the stratum α is defined along the complex linear space $E^{[2]}(\alpha)$, which is transversal to the vector space (60). Therefore this direction of the deformation on α is transversal to the image $Im(f_2)$ of the polyhedron. The deformation $f_2 \mapsto f_3$ itself is defined by linearity along the momentum coordinates on strata of less deeps near α, as at the previous step of the construction. The subcase $(2,3)$ is considered.

In the last cases the deformation $f_2 \mapsto f_3$ is defined analogously to the deformation $f_1 \mapsto f_2$ of the previous step. In the subcases $(2,2)$ and $(2,3)$ the formula of the resulting mapping is obvious.

By the construction, the mapping f_3 has no self-intersection points in a neighborhood of an arbitrary non-admissible pair of strata of the deeps 0 and 1, of the deeps 1 and 2, and of the deeps 0 and 2. Self-intersection points near the considered pairs of strata exist on strata of the deep not less then 3. Evidently, that the mapping f_3 satisfies the condition (C) outside strata of the deeps not less then 3. The deformation $f_2 \mapsto f_3$ is well-defined.

Let us assume that the mapping f_{i-1} is well-defined and the deformation $f_{i-1} \mapsto f_i$ was defined inside δ_i–neighborhood of strata of the deep $i-1$ outside the δ_{i+1}–neighborhood of strata of the deep i.

An auxiliary C^1-infinitesimal deformation $f_{i-1} \mapsto f_i$ is defined on $\Sigma_\circ^{[i]}$, fixed on $\Sigma_\circ^{(i+1)}$. This deformation is along the linear complex space $E(x)^{[i]}$ over elementary strata of the space $\Sigma_\circ^{[i]}$. Then this deformation is extended to a C^0–infinitesimal deformation over the space $\mathbb{RP}^{n-k}$, which is fixed on $\Sigma_\circ^{(i+1)}$.

Assume that α is a stratum of the deep i and a pair of strata $\alpha \prec \beta_1$ is non-admissible. Let us consider all strata $\beta_2, \dots$, which is contained in the closure of the boundary of the stratum β_1, such that the pair with β_1 is non-admissible, and, therefore, the corresponding pair with α is admissible. Let us consider all strata $\beta, \dots$, which is contained in the closure of the boundary of the stratum β_1, such that the pair with β_1 is admissible, and, therefore, the corresponding pair with α is non-admissible. The mapping f_i has no self-intersection points near α in all neighborhoods of non-admissible pairs, on exterior components of the boundaries the mapping f_i coincides with f_{i-1}.

An axillary deformation is constructed. Generally speaking, self-intersection points on strata $\beta_2, \dots$ near the stratum α, such that the corresponding pair with β_1 is admissible, are possible.

Then all the formulas of the deformations of pairs of strata have to be trivialized. At the last step the required mapping f_i, using the formula as in the subcase $(2,4)$ is constructed. The required properties of the mapping f_i follows from the construction. Lemma 6 is proved.

Proof of Lemma $2A$

The mapping into the right space of the diagram (8) defines the required mapping rez in Lemma 4, which satisfies the boundary conditions and determines the reduction of the structural mapping. Lemmas 4 and $2A$ are proved.

5 Proof of Lemma $2B$

Let us prove Lemma 3, assuming the dimensional restrictions (3). Let us construct the classifying space, which will be denoted again by $R\Sigma$.

Let us consider the mapping c'_1 which is defined, assuming dimensional restrictions (3), the same way that in the case of the dimensional restrictions (2), considered above. The mapping c'_1 could have only self-intersection points of multiplicities 2,3, or 4 (because c'_1 is generic, this map have no self-intersection points of the multiplicity 5 and greater).

Let us denote by

$$L \subset N \tag{61}$$

the polyhedron of self-intersection points $[(x,y)]$, for which at least one of the two points $x \in \bar{N}$, or $y \in \bar{N}$ is a critical point of c'_1. Obviously, we have $\partial N \subset L$, let us denote $(\partial N \cap L) \subset L$ by ∂L, and $L \setminus \partial L$ by $L_\circ$. Let us denote by $\bar{L} \subset \bar{N}$ the canonical 2-sheeted cover (with ramifications) over the polyhedron $L \subset N$. Denote by $U(L) \subset N$ a small regular neighborhood of the polyhedron $L \subset N$. In particular, because $\partial N \subset L$, we get the inclusion $\partial N \subset U(L)$.

Denote by $M \subset L$ the polyhedron in L of self-intersection points of the multiplicity 4. Denote by $W_L \subset W_N \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ small equivariant neighborhoods of polyhedra $\bar{L} \subset \bar{N} \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ correspondingly.

Let us prove the following lemma, this lemma in a straightforward analog of Lemma 4.

Lemma 7. *1. There exist a $\mathbb{Z}/2$–equivariant mapping $d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n$, which has a holonomic self-intersection in the sense of Definition 1 and which coincides with the extension of the mapping $i_2 \circ c'_1 : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-4} \subset \mathbb{R}^n$ in a small regular neighborhood of the diagonal.*

2. Moreover, there exists a resolution map $rez : (N \setminus M) \to RK$ to the corresponding spaces of the second row of the diagrams (35). The map $\mu_N = \phi \circ rez : N \setminus M \to K(\mathbf{I}_a, 1)$ is extended to the mapping $\mu_a : N \to K(\mathbf{I}_a, 1)$ and satisfies the boundary conditions over the component ∂N_{diag}.

3. The mapping $\mu_a : N \to K(\mathbf{I}_a, 1)$ satisfies the equations (10) *and induces a cyclic structure of the mapping $d^{(2)}$.*

We shall prove Lemma 3, assuming dimensional restrictions (3).

Configuration space of boundary singularities of the mapping c

Let us consider an ordered pair of points $(x, y) \in (\mathbb{RP}^{n-k})^2$, $x \ne y$, assuming the condition $p(x) = p(y) \in J$, where the mapping p is given by the formula (14). Let us assume that the point x is a singular point of the mapping p. This implies that there exist an (unordered) pair of point $[(x_1, x_2)]$, each the point is infinitesimal-closed to the point x, such that the following conditions are satisfied:

$$x_1 \ne x_2, \quad p(x_1) = p(x_2). \tag{62}$$

Assume that the pair of points $[(x_1, x_2)]$ satisfies the condition (62) and such there exists a point y_1, infinitesimal-closed to the point x, such that the following condition is satisfied:

$$p(y) = p(x_1) = p(x_2). \tag{63}$$

Let us define a space $Z'_{3\circ}$ as the space of all ordered triples of points $((x_1, x_2), y)$, where the both points of the pair (x_1, x_2) are infinitesimal-closed to a point x' and satisfies the condition (62), the point y is infinitesimal closed to y', $p(x') = p(y')$, moreover, the triple $((x_1, x_2), y)$ has to satisfies the condition (63).

The following mapping

$$F_{3\circ} : Z'_{3\circ} \to \Sigma_\circ, \tag{64}$$

which transforms a triple of points $((x_1, x_2), y) \in Z'_{3\circ}$ into a non-ordered pair of points $[(x_1, y)]$ is well defined.

Let us define a space $Z_{4\circ}$ as the space of all non-ordered pairs of ordered pairs of points $[((x_1, x_2), (y_1, y_2))]$, where the points of the non-ordered pair $([x_1, x_2])$ are infinitesimal closed to a point x', a non-ordered pair $[(y_1, y_2)]$ is infinitesimal closed to a point y'. Moreover, we claim that the triples $(x_1, x_2), y_1)$ $(y_1, y_2), x_1)$ satisfy the conditions (62), (63) i.e. this triples are defined the corresponding points of $Z'_{3\circ}$. Additionally, let us assume that the following condition is satisfied:

$$p(y_1) = p(x_1), \tag{65}$$

this condition with the pair of equations (63) implies the equation

$$p(x_1 = p(x_2) = p(y_1) = p(y_2).$$

Let us define the canonical 2-sheeted covering

$$p_{Z_{4\circ}} : \bar{Z}_{4\circ} \to Z_{4\circ}, \quad (66)$$

the fiber of this covering over a non-ordered pair of points $[((x_1, x_2), (y_1, y_2))]$ is defined as ordered pairs of the same points.

The following 2-valued mapping

$$F_{4\circ} : Z_{4\circ} \to Z'_{3\circ}, \quad (67)$$

which transforms a non-ordered pair $[((x_1, x_2), (y_1, y_2))] \in Z_{4\circ}$ into two ordered triples of points $((x_1, x_2), y_1)$, $((y_1, y_2), x_1)$. The following mapping

$$G_{4\circ} = F_{3\circ} \circ F_{4\circ}, \quad (68)$$

which transforms a non-ordered pair of ordered pairs of points $[((x_1, x_2), (y_1, y_2))] \in Z_{4\circ}$ into the non-ordered pair of points $[(x_1, y_1)]$ is well defined.

Let us define the space $Z_{3\circ}$ as the space, which is obtain by the completion of the space $Z'_{3\circ}$ by pairs of points, for which $x_1 = x_2, y, x_1 \neq y$. Let us define a completion Z_3 of the space $Z_{3\circ}$ by points, for which $x_1 = x_2 = y$. The mapping $F_{3\circ}$ is naturally extended to the mapping $F_3 : Z_3 \to \Sigma$.

Space $R\Sigma$ in Lemma 3

Let us define the space $Z_\circ$, which is obtained from $Z_{3\circ}$ by gluing of the space $Z_{4\circ}$ by means of the 2-valued mapping $F_{4\circ}$. The space $Z_\circ$ is a cylinder of (a multivalued) mapping $F_{4\circ}$. The following mapping $F_\circ : Z_\circ \to \Sigma_\circ$ is well defined as the composition of (a multivalued) projection $Z_\circ \to Z_{3\circ}$ with the mapping $Z_{3\circ} \to \Sigma_\circ$. The following diagram is well defined:

$$Y_\circ \subset \Sigma_\circ \stackrel{F_\circ}{\longleftarrow} Z_\circ,$$

where the space $Y_\circ \subset \Sigma_\circ$ is defined by the formula (56). Let us define the space $R\Sigma_\circ$ as the result of gluing of the space $Y_\circ$ with the space $Z_\circ$ along the subspace $F_\circ^{-1}(Y_\circ)$ by the mapping $F_\circ|_{F_\circ^{-1}(Y_\circ)}$. The space $R\Sigma$ is defined analogously, by replace the space $\Sigma_\circ$ to the space Σ and the space $Z_{3\circ}$ to the space Z_3 in the construction. The space $R\Sigma$ is obtained from the space $R\Sigma_\circ$ using the completion by Q_{diag}.

Resolution mapping $\phi_\circ : R\Sigma_\circ \to K(\mathbf{I}_a, 1)$

The mapping $\phi_\circ|_{Y_\circ}$ on the subspace $Y_\circ \subset R\Sigma_\circ$ was constructed in the proof of the statement A of the lemma. Let us construct the mapping

$$\phi_{Z_\circ} : Z_\circ \to K(\mathbf{I}_b, 1). \tag{69}$$

Then let us prove that the mapping $i_{\mathbf{I}_b,\mathbf{D}} \circ \phi_{Z_\circ} : Z_\circ \to K(\mathbf{I}_b, 1) \subset K(\mathbf{D}, 1)$ coincides with the composition $\eta_\circ \circ F_\circ : Z_\circ \to K(\mathbf{D}, 1)$, where $\eta_\circ$ is the structural mapping, which is given by the formula (23).

Let us define the mapping

$$\phi_{Z'_{3\circ}} : Z'_{3\circ} \to \mathbb{RP}^{n-k} \subset K(\mathbf{I}_d, 1) \tag{70}$$

by the formula $\phi_{Z'_{3\circ}}((x_1, x_2), y) = y$. The mapping (70) is expendable to the following mapping

$$\phi_{Z_{3\circ}} : Z_{3\circ} \to \mathbb{RP}^{n-k} \subset K(\mathbf{I}_d, 1). \tag{71}$$

Lemma 8. *By the assumption* $r \equiv 0 \pmod 2$ *the mapping* $\eta_\circ \circ F_{3\circ} : Z'_{3\circ} \to \Sigma_\circ \to K(\mathbf{D}, 1)$ *allows a reduction to the mapping in the subspace* $K(\mathbf{I}_d, 1) \subset K(\mathbf{D}, 1)$*, which is homotopic to the mapping* (71)*.*

Proof of Lemma 8

Let $m : S^1 \subset Z_{3\circ}$ be a path, which is projected into the path $l : S^1 \subset U(Q_{diag})_\circ \subset K_\circ$ by means of the mapping $F_{3\circ}$. Let us assume that the path l transversely intersects the subspace $K_\circ^{[1]}$ of singular strata of the deep 1, and does not intersect strata of the deep 2 of the stratification (48). Let us denote intersection points by $\{t_1, \dots, t_j\}$.

On each elementary stratum of the deep 0 of the polyhedron $U(Q_{diag})_\circ$, which contains the point $pt = l(0)$, $0 \in S^1$ is the market point, let us fixes a coordinate system $(\check{x}_1(0), \check{x}_2(0))$.

Let us extend in a natural way a coordinate system $(\check{x}_1(0), \check{x}_2(0))$ from the point pt along the path m (about the natural continuation coordinate system in neighborhoods of points $\{t_1, \dots, t_j\}$ see the construction of the structural maps $\eta_\circ$).

The transformation of the initial system of coordinates $(\check{x}_1(0), \check{x}_2(0), \check{x}_3(0))$ in the system of coordinates $(\check{x}_1(2\pi), \check{x}_2(2\pi), \check{x}_3(2\pi))$ is well defined. This transformation is represented by the direct product of two natural transformations $\eta_{1,2} \in \mathbf{I}_d$, $\eta_3 \in \mathbf{I}_d$ of the first two and the third coordinate respectively.

The homomorphism $\phi_{*,Z_{3\circ}}(m)$, which is associated with (71) is defined by the formula $\phi_{*,Z_{3\circ}}(m) = \eta_3(m)$. The homomorphism $(\eta_\circ \circ F_\circ)_*(m)$ is defined by the formula $(\eta_\circ \circ F_\circ)_*(m) = i_{\mathbf{I}_d,\mathbf{D}} \circ \eta_{1,2}(l)$. Let us prove the following equation:

$$\eta_3(m) = \eta_{1,2}(l). \tag{72}$$

Let us assume that the path l is inside the only elementary stratum of the deep 0. In this case the equation (72) is evident.

Let us consider a general case. Let us define a path l', which is homotopic to the path l in $U(Q_{diag})_\circ$, and satisfies the following conditions. For an arbitrary point $a \in l'$ let us define the principle and the auxiliary coordinate, such that the principle coordinate have the residue $+1$ and this coordinate corresponds to the regular momentum coordinate. Each auxiliary coordinate has a residue -1, this coordinate corresponds to the singular momentum coordinates of an infinitesimal closed point $a' \in Q_{diag}$.

Let the path l' does not intersect strata of the deep 2, let $\{x_1, \dots, x_s\}$ be the finite set of the critical values of the path l, in which point the path crosses the stratum of the depth 1. Each critical value x_i of the path l determines a pair of closed critical values y_i, z_i of the path l'. It is required that l' on the interval (x_{i-1}, x_i) has the only one axillary coordinate, the number of this coordinate denote by $j([i-1,i])$. It is required that the path l' on the short interval (y_i, z_i) has the two axillary coordinates, which is denoted by $j([i-1,i])$ и $j([i,i+1])$.

Moreover, in the neighborhood of each critical points of the path l' the following conditions are satisfied (which is not a restriction of a generality):

– At the critical points the collection of the principal and the axillary coordinates are continuous;

– At the point y_i the first axillary coordinate with numbers $j([i-1,i])$, $j([i,i+1])$ change continuously, the second axillary coordinate with the number $j([i,i+1])$ is not continuous and is changed to the opposite, the second axillary coordinate with the number $j([i-1,i])$ is continuous (this condition implies that on the short interval (y_i, z_i) the residues of the first two axillary coordinates is equal to -1); the third auxiliary coordinate is continuous;

— At the point z_i the first two axillary coordinate with numbers $j([i-1,i])$, $j([i,i+1])$ change continuously; the second axillary coordinate with the number $j([i-1,i])$ is not continuous and is changed to the opposite, the second axillary coordinate with the number $j([i,i+1])$ is continuous (and therefore on the interval (z_i, y_{i+1}) the residue of the first two axillary coordinates is equal to -1), the third axillary coordinate with the number $j([i,i+1])$, generally speaking, is not continuous and is changed by the multiplication on the element

$$\theta(x_i) \in \mathbf{I}_d \tag{73}$$

with the prescribed value of the turd coordinate at the point x_i along the path l'.

Evidently, the homotopies $l \mapsto l'$ $m \mapsto m'$ are well defined, where $F_{3\circ}(m) = l$, $F_{3\circ}(m') = l'$.

Let us prove that the value $\eta_3(m') \in \mathbf{I}_d$ is given by the following formula:

$$\eta_3(m') = \prod_i \theta(x_i). \tag{74}$$

For the path m' let us consider another path m'_0, which is also projected onto l' and satisfies the equation $\theta(x_i) = 1$ for an arbitrary i. The homology class of the path $\eta_3(m')$ is changed from the homology class of the path $\eta_3(m'_0)$ by the element $\prod_i \theta(x_i)$ and, moreover, the following formula (72) for the path m'_0 is evident.

For the path m' the following equation is satisfied:

$$\prod_i \theta(x_i) = 1.$$

The proof follows from the following facts: the number of jumps of coordinates coincide; the number r of coordinates is even. Lemma 8 is proved.

Let us consider the involution on the canonical covering $\bar{Z}_{4\circ}$, this covering is given by the formula (66). The involution transforms the ordered triple $((x_1, x_2), (y_1, y_2))$ into the ordered triple $((y_1, y_2), (x_1, x_2))$. Let us consider the composition $\phi_{Z_{3\circ}} \circ F_{4\circ} \circ p_{Z_{4\circ}} : \bar{Z}_{4\circ} \to Z_{4\circ} \to Z'_{3\circ} \to K(\mathbf{I}_d, 1)$. It is not difficult to prove that this composition is factorized to the following mapping:

$$\phi_{Z_{4\circ}} : Z_{4\circ} \to K(\mathbf{I}_d, 1). \tag{75}$$

From the Lemma 8 we get, that the mapping $\phi_{Z_{4\circ}} \circ p_{Z_{4\circ}} : \bar{Z}_{4\circ} \to K(\mathbf{I}_d, 1)$ coincides with the mapping $\phi_\circ \circ G_{4\circ} \circ p_{Z_{4\circ}} = \phi_{Z_{3\circ}} \circ F_{4\circ} \circ p_{Z_{4\circ}} : \bar{Z}_{4\circ} \to K(\mathbf{I}_d, 1)$. Therefore we get $\phi_\circ \circ G_{4\circ} = \phi_{Z_{3\circ}} \circ F_{4\circ}$. Therefore the resulting mapping $\phi_\circ : R\Sigma_\circ \to K(\mathbf{I}_a, 1)$ is well defined.

Proof of Lemma 3

The space $R\Sigma$ and the resolution mapping is defined above. For a suitable mapping (75) (this mapping is well defined up to the composition with the standard mapping $K(\mathbf{I}_b, 1) \to K(\mathbf{I}_b, 1)$, given by the conjugation, the boundary conditions are satisfied. Lemma 3 is proved.

Proof of Lemma 7

Let us prove the statement 1. Assuming the dimensional restrictions (3), the mapping $c'_1 : \mathbb{RP}^{n-k} \to \mathbb{R}^{n-4}$ is not an embedding and could have self-intersection points of multiplicities $2, 3$, or 4. Let us consider the diagram (35). Denote by $N2_\circ \subset N'_\circ$ the open polyhedron of self-intersection regular points of c'_1. This polyhedron is immersed into $\mathbb{R}^{n-4}$. By the definition the following decomposition is well defined: $N2_\circ = N'_\circ \setminus L_\circ$, where $L_\circ$ is the obvious denotation for the polyhedron $L \setminus (L \cap \partial N)$, L is defined by the formula (61). Let us denote by $N2'_\circ \subset N2_\circ$ a little smaller polyhedron, which is defined by the cutting out from $N2_\circ$ the regular neighborhood of the polyhedron $L_\circ$. The following classifying mapping

$$U(N')_\circ \setminus M) \to R\Sigma_\circ \qquad (76)$$

is well defined.

Let us construct the equivariant deformation of the equivariant mapping $(c'_1)^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^{n-4} \times \mathbb{R}^{n-4} \subset \mathbb{R}^n \times \mathbb{R}^n$ with the support in a small regular neighborhood of the subpolyhedron $N2'_\circ \times N2'_\circ \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ to the required equivariant mapping $d^{(2)}$ in the space of formal mappings with holonomic self-intersections. In the considered neighborhood of $N2'_\circ$ the mapping c'_1 is an immersion, therefore by Lemma 6 a formal regular deformation, which deforms self-intersection polyhedron N' into self-intersection polyhedron N is well defined. The mapping $rez : N \setminus M \to R\Sigma$ is well defined by an extension of the mapping (76). This mapping satisfies the prescribed boundary conditions on the diagonal. The statement 1 is proved.

Let us prove the statement 2. The polyhedron $M \subset L$ by general position arguments satisfies the following condition: in each simplex of the triangulation of the polyhedron L the corresponding subsimplex of the subpolyhedron M has at least the codimension 3. Therefore the mapping $\phi \circ rez : N \setminus M \to K(\mathbf{I}_a, 1)$, constructed above, is extended uniquely up to homotopy to a mapping on the hole N. The mapping (9) is well defined. The statement 2 is proved.

Let us prove the statement 3. It is sufficiently to check that the equations (10) are satisfied (see also [A1,(41)]). Let us prove this equation in the case $q = 0$. For an an arbitrary positive q the proof is analogous.

Use the Lemma 27 in [A1], whereby it is sufficient to consider the absolute cycle $\mu_{\bar{R}} : \bar{R}^{n-k} \to K(\mathbf{I}_d, 1)$ (recall that this cycle is obtained by means of the gluing of the map $\bar{\mu}_{a;N} : (\bar{N}, \partial \bar{N}) \to K(\mathbf{I}_d, 1)$ along the boundary by a cylinder) and it is sufficient to verify that the cycle $\mu_{\bar{R}}$ defines a generator of the homology group $H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$. We denote the homology class of the cycle $\mu_{\bar{R}}$ by $x \in H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$.

Consider another homology class $p_{\mathbf{I}_c,\mathbf{I}_d;*} \circ \bar{\eta}_*([\bar{N}]) \in H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$, which is determined by the cycle, obtained by means of the composition of the canonical covering $\bar{\eta} : \bar{N} \to K(\mathbf{I}_c, 1)$ over the characteristic map $\eta : N \to K(\mathbf{D}_4, 1)$ with the map $p_{\mathbf{I}_c,\mathbf{I}_d} : K(\mathbf{I}_c, 1) \to K(\mathbf{I}_d, 1)$. In the last formula the polyhedron $\bar{N}$ is considered as a closed manifold, i.e. the canonical covering is taken first over $N_\circ$, where this canonical covering is well defined. Then the standard compactification is defined. The result of this standard compactification is defined as the total space of the covering with ramification over the boundary ∂N. The manifold $\bar{N}$ is closed, i.e. the canonical covering is branched over the boundary ∂N. Denote this homology class by $y \in H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$.

Let us prove that y is a generator. Indeed, the cohomology class of y^{op} that is Poincaré dual to the homology class y is calculated as a normal characteristic class $\bar{w}_k \in H^k(\mathbb{RP}^{n-k}; \mathbb{Z}/2) = H^k(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$. (When $n = 2^\ell - 1$ the cohomology class $\bar{w}_k$ is nontrivial.)

Consider another homology class $p_{\mathbf{I}_c,\mathbf{I}_d;*} \circ \bar{\eta}_*([\bar{N}]) \in H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$, which is determined by the cycle, obtained by means of the composition of the canonical covering $\bar{\eta} : \bar{N} \to K(\mathbf{I}_c, 1)$ over the characteristic map $\eta : N \to K(\mathbf{D}_4, 1)$ with the map $p_{\mathbf{I}_c,\mathbf{I}_d} : K(\mathbf{I}_c, 1) \to K(\mathbf{I}_d, 1)$. In the previous formula the polyhedron $\bar{N}$ is considered as a closed manifold: the canonical covering is taken first over $N_\circ$, where this covering is well defined. Then the standard compactification is defined as the total space of the branched covering over the boundary ∂N. Denote this homology class by $y \in H_{n-k}(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$.

Let us prove that y is a generator. The cohomology class of y^{op} that is Poincaré dual to the homology class y is calculated as a normal characteristic class $\bar{w}_k \in H^k(\mathbb{RP}^{n-k}; \mathbb{Z}/2) = H^k(K(\mathbf{I}_d, 1); \mathbb{Z}/2)$. (When $n = 2^\ell - 1$ the cohomology class $\bar{w}_k$ is nontrivial.)

Now let us check that $x = y$. Recall that $N = N_K \cup N_{\Sigma_{antidiag}}$, $\partial N_K = (N_{antidiag} \cup N_{diag}) \subset N_K$. Therefore, the 2-sheeted covering $\bar{N}$ itself is represented by the following union: $\bar{N} = \bar{N}_{\Sigma_{antidiag}} \cup_{\bar{N}_{antidiag}} \bar{N}_K$. The following analogous formula is satisfied: $\bar{R} = \bar{L}(d_0) \cup_{\bar{R}_{antidiag}} \bar{R}_{K_0}$. Moreover, the mappings $\bar{\mu}_{a;N(d_0)}$ and $\mu_{\bar{R}}$ on the common part $\bar{L}(d_0)$ coincide, because the mappings on the antidiagonal part are cyclic. The following formula is satisfied: $\bar{R} = \bar{R}_{\Sigma_{antidiag}} \cup_{\bar{R}_{antidiag}} \bar{R}_K$. We have $N_{\Sigma_{antidiag}} = R_{\Sigma_{antidiag}}$. Moreover, the mappings $\bar{\mu}_a$ and $\mu_{\bar{R}}$ on the common part $\bar{N}_{\Sigma_{antidiag}}$ coincide.

Let us use an additional symmetry on the singular polyhedron. The involution on N_{K° is well defined. This involution is induced by the involution (46). This involution is invariant on the diagonal and the antidiagonal and is free outside the antidiagonal component of the boundary. The homology class $x - y$ is represented by the cycle $\bar{\mu}_a|_{\bar{N}} \cup \bar{\mu}_R|_{\bar{R}_K} : \bar{N}_K \cup \bar{R}_K \to K(\mathbf{I}_d, 1)$, this cycle is homologous to zero. Lemmas 2 and 7 are proved.

6 Application

Let us prove [Proposition 28,A1] with the dimensional restriction (3). The proof of this proposition is based on [Lemma 14, A3], this lemma contains a mistake (self-intersection points of the multiplicity 4 are not well considered and the mapping t_0 does not extended on the polyhedron of self-intersection points). Additionally [A3] contains no proof of Lemma 23. Definition 24 in [A1] of cyclic structure for mappings with singularities has to be changed, the correction is in the presented paper.

Proof [Proposition 28,A1] by means of Lemma 2

Let us consider the equivariant mapping $d^{(2)} : \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k} \to \mathbb{R}^n \times \mathbb{R}^n$, constructed in Lemma 2. The mapping $d : \mathbb{RP}^{n-k} \to \mathbb{R}^n$ is well defined, the formal extension of this mapping coincides with the mapping $d^{(2)}$ near the diagonal. Denote by $i_{\bar{U}_N} : \bar{U}_N \subset \mathbb{RP}^{n-k} \times \mathbb{RP}^{n-k}$ the equivariant neighborhood, which is rather far from the diagonal, where the mapping $d^{(2)}$ is holonomic.

Define the open manifold V^{n-k} of the dimension $n-k$ as the minimal manifold, for which the standard embedding $i_{\bar{U}_N} : \bar{U}_N \subset V^{n-k} \times V^{n-k}$ is well-defined. Let us consider the immersion $\phi_V : V^{n-k} \looparrowright \mathbb{RP}^{n-k}$, for which the composition $\phi_U \circ p_{\bar{U}_N} : \bar{U}_N \to V^{n-k} \to \mathbb{RP}^{n-k}$, where $p_{\bar{U}_N} : \bar{U}_N \to V^{n-k}$ is the natural projection onto the first factor, coincides with the projection $Im(i_{\bar{U}_N})$ on the first factor.

Denote by $U_{\partial N} \subset \mathbb{RP}^{n-k}$ the regular ε_1–neighborhood of critical points of the mapping d. The image of the restriction of the mapping ϕ_V on $V^{n-k} \cap \phi_U^{-1}(U_{\partial N})$ belongs to the neighborhood $U_{\partial N} \subset \mathbb{RP}^{n-k}$.

By assumption the mapping $d^{(2)}|_{\bar{U}_N} : \bar{U}_N \to \mathbb{R}^n \times \mathbb{R}^n$ has a holonomic self-intersection and the mapping $\theta : V^{n-k} \to \mathbb{R}^n$, such that $d^{(2)}$ is the restriction of the extension of the mapping θ on $Im(i_{\bar{U}_N})$ is well defined.

The restriction of the mapping θ on $V^{n-k} \cap \phi_U^{-1}(U_{\partial N})$ coincides with the composition $d \circ \phi_V$. The marked component (possibly, non-connected) of the self-intersection polyhedron of the mapping θ is well defined, this component is PL-homeomorphic to the polyhedron N outside $U_{\partial N}$.

Let us consider an arbitrary element of the group $Imm^{sf}(n-k,k)$, this element is represented by an immersion $f' : M^{n-k} \looparrowright \mathbb{R}^n$, equipped with a skew-framing (κ, Ξ'). Let us consider the characteristic class $\kappa : M^{n-k} \to \mathbb{RP}^{n-k}$ of this skew-framing.

Consider the open neighborhood $U_{\partial N} \subset \mathbb{RP}^{n-k}$ and denote by $UM^{n-k} \subset M^{n-k}$ the inverse image $\kappa^{-1}(U_{\partial N})$ of this neighborhood. Consider an open subdomain $Im(\phi_V) \subset \mathbb{RP}^{n-k}$ and denote by $VM^{n-k} \subset M^{n-k}$ the inverse image $\kappa^{-1}(Im(\phi_V))$ of this subdomain. The immersion $\phi_V : V^{n-k} \looparrowright Im(\phi_V) \subset \mathbb{RP}^{n-k}$ induces the immersion $\phi_{WM} : WM^{n-k} \looparrowright VM^{n-k} \subset M^{n-k}$ and the mapping $\kappa_{WM} : WM^{n-k} \to V^{n-k}$ by natural way.

Consider the subdomain $VM^{n-k} \cap UM^{n-k} \subset M^{n-k}$ and denote by VUM^{n-k} the inverse image $\phi_{WM}^{-1}(VM^{n-k} \cap UM^{n-k}) \subset WM^{n-k}$. The composition $\kappa \circ \phi_{WM} : WM^{n-k} \to VM^{n-k} \to \mathbb{RP}^{n-k}$ is such that the restriction of this composition onto the domain $VUM^{n-k} \subset WM^{n-k}$ coincides with the composition $\kappa \circ \phi_{WM} : VUM^{n-k} \to VM^{n-k} \cap UM^{n-k} \to \mathbb{RP}^{n-k}$.

Consider the mapping $\theta \circ \kappa_{WM} : WM^{n-k} \to V^{n-k} \to \mathbb{R}^n$ and the mapping $d \circ \kappa : UM^{n-k} \to U_{\partial N} \to \mathbb{R}^n$. These two mappings coincides on the subdomain $VUM^{n-k} \subset WM^{n-k}$, i.e. the restrictions $\theta \circ \kappa_{WM}$ and $d \circ \kappa \circ \phi_{WM}$ to this domain coincide.

Let us consider the immersion $f' : M^{n-k} \looparrowright \mathbb{R}^n$. Let us apply C^0–principle [Preposition 30,A1] and let us construct an immersion $\alpha_1 : WM^{n-k} \looparrowright \mathbb{R}^n$, ε–closed to the composition $\theta \circ \kappa_{WM}$ and the immersion $\alpha_2 : UM^{n-k} \looparrowright \mathbb{R}^n$, ε–closed to the composition $d \circ \kappa$, where $\varepsilon << \varepsilon_1$ and satisfy the following properties:

–1. the immersions α_1, α_2 coincide on the common subdomain $VUM^{n-k} \subset WM^{n-k}$, i.e. the restrictions of α_1 and $\alpha_2 \circ \phi_{WM}$ onto this subdomain coincide;

–2. the immersion α_1 is in the regular homotopy class of $f' \circ \phi_{WM}|_{WM^{n-k}}$, the immersion α_2 is in the regular homotopy class of $f'|_{UM^{n-k} \subset M^{n-k}}$.

Self-intersection points of the immersion α_1 is the manifold with boundary, denote this manifold by $N_{\alpha_1}^{n-2k}$; Self-intersection points of the immersion $\alpha_2 \circ \phi_{WM}$ is also the manifold with boundary, denote this manifold by $N_{\alpha_2}^{n-2k}$. The constant ε is small enough, therefore the manifold $N_{\alpha_2}^{n-2k}$ contains the component which is diffeomorphic to the neighborhood of the boundary of the component $N_{\alpha_1}^{n-2k}$ and is denoted by $NWU_{\alpha_2}^{n-2k} \subset N_{\alpha_2}^{n-2k}$. Moreover the following two properties are satisfied.

The components of the manifold $N_{\alpha_2}^{n-2k} \setminus NWU_{\alpha_2}^{n-2k}$ are divided into the following two types. The component of type 1 is immersed into a regular neighborhood of the self-intersection polyhedron of the mapping d. All the last self-intersection points of α_2 belong to the components of type 2. Note that the component of the type 2 consists of self-intersection points $x_1 \in M^{n-k}, x_2 \in M^{n-k}$, $\alpha_2(x_1) = \alpha_2(x_2)$, for which $\kappa(x_1)$ and $\kappa(x_2)$ are closed on $\mathbb{RP}^{n-k}$. Let us take the union of the component of type 2 with $NWU^{n-2k}_{\alpha_2}$ and denote this union by $N0^{n-2k}_{\alpha_2}$.

The components $N^{n-2k}_{\alpha_1}$, $N0^{n-2k}_{\alpha_2}$ are glued together into one component: $N0^{n-2k}_{\alpha_2} \cup_{NWU^{n-2k}_{\alpha_2}} N^{n-2k}_{\alpha_1}$. By the construction the manifold $NWU^{n-2k}_{\alpha_2}$ is ε–closed immersed into ε_1–regular neighborhood of the boundary of the self-intersection polyhedron N of the mapping $\theta : V^{n-k} \to \mathbb{R}^n$.

The manifold $N0^{n-2k}_{\alpha_2} \cup N^{n-2k}_{\alpha_1}$ is equipped by an immersion into $\mathbb{R}^n$, let us denote this immersion by g. The immersion g is a $\mathbf{D}$–framed immersion, because the manifold $N0^{n-2k}_{\alpha_2} \cup N^{n-2k}_{\alpha_1}$ is a component of self-intersection manifold of a skew-framed immersion. The self-intersection manifold is presented as an intersection of the two regular sheets. In particular, an immersion α'_2, for which $N0^{n-2k}_{\alpha_2}$ is a closed component of intersection is well defined and, moreover, one may assume that the immersion α'_2 coincides with the immersion α_1 in a domain, which self-intersects along $NWU^{n-2k}_{\alpha_2}$.

The self-intersection manifold of the immersion $\alpha'_2 \cup \alpha_1$ of the manifold with boundary contains a closed component $N0^{n-2k}_{\alpha_2} \cup_{NWU^{n-2k}_{\alpha_2}} N^{n-2k}_{\alpha_1}$. Denote the parametrized immersion of this component by g, denote the $\mathbf{D}$–framed of this immersion by Ψ.

The Hopf invariant $h(f', \kappa, \Xi')$ of the skew-framed immersion (κ, Ξ') coincides with the degree modulo 2 of the mapping κ. The Hopf invariant of the framed immersion (g, Ψ) coincides with $h(f', \kappa, \Xi')$.

Evidently, in the regular cobordism class $[(f'\Xi')]$ there exists a skew-framed immersion (f, Ξ), which has a closed component of the self-intersection manifold, which is diffeomorphic to $N0^{n-2k}_{\alpha_2} \cup N^{n-2k}_{\alpha_1}$ and is parametrized by a $\mathbf{D}$-framed immersion (g, Ψ). The mapping $\mu_a : N0^{n-2k}_{\alpha_2} \cup N^{n-2k}_{\alpha_1} \to K(\mathbf{I}_a, 1)$ is well defined. This mapping is define on the submanifold $N0^{n-2k}_{\alpha_2}$ by means of the cyclic structure of $d^{(2)}$ and on the submaniflod $N^{n-2k}_{\alpha_1}$ by the mapping κ. The mapping μ_a on all last components of self-intersection of the immersion g define as the trivial mapping. The skew-framed immersion (f, Ξ) is equipped with the cyclic structure, described above. [Proposition 28,A1] is proved.

Remark

The statement of [Lemma 13,A3] contains a mistake. This mistake admits the analogous correction. The definition of abelian structure, used in [Lemma 1,A2] has to be changed and formulated analogously to the definition of cyclic structure in the presented paper. Analogous changes has to be considered in [Definitions 28, 29, A2].

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142190 Moscow Region, Troitsk, IZMIRAN
pmakhmet@izmiran.ru