Toric Degenerations and the Exact Bohr-Sommerfeld Correspondence

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Outline.

1. Background: B-S conditions
2. Toric Integrable Systems
3. Singular Toric Systems
4. Relation to Geodesic Flow

The topics discussed here come from joint work with Alejandro Uribe, and earlier work with Uribe, V. Guillemin and Z. Wang.

1. Background

Geometric quantization seeks to associate a quantization (space of wave functions, operators, etc.) to a classical mechanical system. One is given

- a symplectic manifold M, ω
- A line bundle L on M with connection ∇, such that

$$c_1(L, \nabla) = \frac{1}{2\pi i}\text{Curvature}(\nabla) = \omega.$$

- A polarization = integrable distribution $\mathcal{F}$ of Lagrangian tangent subspaces of $TM \otimes \mathbb{C}$

We refer to the two extreme cases:

1. The *real* case: $\mathcal{F} = \mathcal{F}_0 \otimes \mathbb{C}$, where $\mathcal{F}_0 \subset TM$

2. The *complex* case: M, ω is Kähler, L is holomorphic, ∇ is the canonical (Chern) connection and $\mathcal{F} = T^{(0,1)}M$

The space of wave functions should be $W = \{\text{sections } s \text{ of } L | \nabla_\xi s = 0, \forall \xi \text{ tangent to } \mathcal{F}\}$.

For the real case, wave functions $W_{\mathbb{R}}$ are sections of L covariant constant along the leaves Λ of $\mathcal{F}$.

For the complex case $W_{\mathbb{C}} = H^0(M, \mathcal{O}(L))$, the space of holomorphic sections of L.

If this has (hopefully) anything to do with physics, one expects/requires the spaces of wave functions to be *independent of polarization*, and ideally there should be a correspondence relating the operators, etc., of two such representations.

The most explicit case is that of a completely integrable system $\mathcal{H}$ on a Kähler manifold M, ω with positive line bundle L as above. We have a set of generating Hamiltonians

$$H_1, \ldots, H_n, n = \dim_{\mathbb{C}} M,$$

defining the Lagrangian foliation by their level sets, and suitably normalized, the quantized, or Bohr-Sommerfeld levels, are given by

$$H_i \in \mathbb{Z}, i = 1, \ldots, n.$$

The line bundle with connection (L, ∇) has curvature zero restricted to each level of $\mathcal{H}$. Bohr-Sommerfeld levels are precisely those where a global flat section of L exists on the (connected) level. Such is unique up to non-zero scaling, and we call it s_Λ, if Λ is a Bohr-Sommerfeld level. Thus, in first approximation, the real quantization of M, L is

$$W_{\mathbb{R}} = \oplus_{\Lambda \in B.S.} \mathbb{C} \cdot s_\Lambda,$$

where we have not stated the Hilbert space structure.

Following Andrei Tyurin, we say that the ***numerical*** Bohr-Sommerfeld correspondence holds if

$$\dim_{\mathbb{C}} W_{\mathbb{R}} = \dim_{\mathbb{C}} W_{\mathbb{C}}.$$

and we say that a linear map

$$B : W_{\mathbb{R}} \to W_{\mathbb{C}}$$

gives an ***exact*** Bohr-Sommerfeld correspondence if it is an isomorphism.

Since $W_{\mathbb{R}}$ is a set of distributional sections of L, there is a natural candidate for a linear operator B given by the Bergman kernel projecting distributional sections of L onto the subspace of holomorphic sections.

Tyurin supposed that this map did display an exact B-S correspondence in great generality, but the theorem of Borthwick-Paul-Uribe cited as proof is (1) about asymptotic behavior of sections of $L^{\otimes N}$, as $N \to +\infty$, and (2) is an argument about amplitudes of sections. Amplitudes may still suffice to show the exact B-S property for B, but it presently seems to depend on information on phases.

We discuss special cases of this correspondence here. In particular, we show that exact BS as formulated here holds for toric varieties, and more interestingly, for singular (projective) toric varieties. We give a degeneration argument which shows that this latter case allows one to verify exact BS for a number of singular integrable systems on Kähler varieties, such as the classical Gelfand-Tsetlin system on the flag manifold, and systems related to the geodesic flow on the complex quadrics, and speculatively, on all compact rank one symmetric spaces (CROSSes). Some open problems will be mentioned along the way.

2. Toric Systems.

A toric system is a Hamiltonian action of a torus T^n on M, ω, which lifts to the line bundle L. It is holomorphic if the action is by holomorphic transformations of M. If $H_1, \ldots, H_n$ are generating Hamiltonians for the torus action, such H_i are then $\mathcal{C}^\omega$ on M.

Let $\mu = (H_1, \ldots, H_n) : M \to \mathfrak{t}^*$, the dual of the Lie algebra of T^n, be the moment map of the action. Recall that, suitably normalized, its image is a polytope Δ with integer vertices. Here a level Λ of μ is Bohr-Sommerfeld iff $H_i|_\Lambda \in \mathbb{Z}, \forall i$, that is

$$\text{B-S levels} \leftrightarrow \Delta_{\mathbb{Z}} = \Delta \cap \mathfrak{t}^*_{\mathbb{Z}}.$$

Theorem 1. B is an isomorphism

$$B : \oplus_{m \in \Delta_{\mathbb{Z}}} \mathbb{C}\Lambda_m \to H^0(M, \mathcal{O}(L)).$$

Proof. Every $m \in \Delta_{\mathbb{Z}}$ determines a character χ_m of T^n. We note the following simple facts:

- s_{Λ_m} is a (distributional) eigensection of L for T^n with character χ_m;

• The line bundle L has a section s_m, unique up to scale, such that s_m transforms under character χ_m on $T^n_{\mathbb{C}}$;
• m is in the interior F^o of a unique face $F \subset \Delta$, and $s_m \neq 0$ on $\mu^{-1}(F^o) \subset M$. In particular, $s_m \neq 0$ on $\mu^{-1}(m) = \Lambda_m$.
• B is T^n-equivariant.

Putting these together, we have by equivariance, $B(s_{\Lambda_m}) = c_m s_m$, for some constant $c_m \in \mathbb{C}$. To show $c_m \neq 0$, we note that

$$\langle B(s_{\Lambda_m}), s_m \rangle = c_m \|s_m\|^2,$$

and

$$\langle B(s_{\Lambda_m}), s_m \rangle = \int_{\Lambda_m} b_m d\lambda_m \neq 0,$$

where b_m is a non-zero constant, and λ_m is invariant measure on Λ_m, which is T^n-homogeneous.

□

This was noted by DB, Guillemin and Uribe, and subsequently by M. Hamilton, independently.

In the above, our M was non-singular and the action of T^n was holomorphic. We can relax the condition that the action of T^n be holomorphic, and then Delzant showed that $M, \omega, \mathcal{H}$ is symplectomorphic to a holomorphic system as above. This is characterized by a Δ which is a "Delzant polytope" in such cases. However, for every convex polytope Δ with integer vertices, we can construct a polarized, normal toric variety, which for $L >> 0$ will be projectively embedded. Such a variety has a natural torus action, and by pulling back the Fubini-Study form from $\mathbb{P}^N$ we can give Hamiltonians for the action such that the moment polytope is this same Δ.

Theorem 1bis. Theorem 1 also holds for a singular, projective toric variety.

Proof. The same proof works. The theory of toric varieties still gives that $s_m \neq 0$ on $\mu^{-1}(F^o)$ as above. The L^2 structure comes from $\frac{1}{n!}\omega^n = \frac{1}{n!}(\omega_{FS}|_M)^n$.

3. Singular Toric Systems.

We now consider systems where we have an integrable system with $\mathcal{C}^\omega$ Hamiltonians, but for which the torus action generators are not $\mathcal{C}^\omega$. Here are some simple examples.

1. The simplest example is to consider the unit sphere $M = S^2 \subset \mathbb{R}^3$ with the usual metric and its oriented area form, and let the Hamiltonian $H = |z|$, where z is the vertical coordinate. The flow of H is periodic on $M \setminus \{z = 0\}$, and on $\{z = 0\}$ the flow is indeterminate. The flow rotates counterclockwise around both the north pole N and the south pole S. The moment polytope is the interval $[0, 1]$ with all points covered with multiplicity 2 except 0, which has multiplicity 1. In particular, unlike the Delzant case, $\mu^{-1}(\zeta)$ is not connected, $\zeta \in (0, 1]$. Note that $z^2 \in \mathcal{C}^\omega(M)$ generates the same integrable system.

2. The classical versions of the Gelfand-Tsetlin systems on complex flag manifolds are of this sort (Guillemin-Sternberg). The generators of the torus action are algebraic functions (solutions of algebraic equations over $\mathcal{C}^\omega$ functions), and the torus action is not defined on special real analytic sets.

3. Let K be a compact Zoll manifold, and let $M = T^*K//_{r=r_0}S^1$ be the symplectic cut of T^*M by the geodesic flow at $r = r_0$, where r is the length function. M is a compact, symplectic manifold with a periodic Hamiltonian r induced from r on T^*M. Example 1 above is the case $K = S^1, r_0 = 1$. (M. Audin considered such symplectic cuts recently.) For the standard examples of Zoll manifolds, the so-called CROSSes, the cuts above can be identified with Hermitian symmetric spaces of rank 2, and we find Gelfand-Tsetlin-like systems there showing the complete

integrability of geodesic flow, sometimes by multiple inequivalent systems.
In examples such as the Gelfand-Tsetlin systems, there cannot be a smooth Hamiltonian T^d action on the flag manifold $\mathbb{F}^n$, $d = \dim_{\mathbb{C}} \mathbb{F}^n$, but the Gelfand-Tsetlin has, *a posteriori*, a convex moment polytope Δ for the image of μ, which is only an algebraic map. Many authors – Kogan-Miller and Alekseev-Brion, for example – have constructed "flat projective degenerations" of such varieties to the *singular* toric variety determined by Δ, and similarly for other spherical varieties with integrable systems. A. Knutson was the first we know of who claimed that in some cases the integrable system would also degenerate to a toric system.
Nishinou-Nohara-K. Ueda have recently used the toric degeneration of the Gelfand-Tsetlin system to prove very interesting results about the potential function in Lagrangian Floer theory for a Lagrangian fiber of the Gelfand-Tsetlin

system. Howard-Millson have discussed degeneration of the polygon space system. Here are some definitions and set-up.

a.) A *flat degeneration* of X_1 to a toric variety X_0 will be a flat and proper holomorphic map $\pi : \mathcal{X} \to \mathcal{B}$ of connected varieties, where 0 and b_1 are two points of $\mathcal{B}$, and we are given identifications of $X_0 = \pi^{-1}(0)$ and a toric variety with polytope Δ, and $\pi^{-1}(b_1)$ with X_1. We have a line bundle $\mathcal{L}$ on $\mathcal{X}$. Set $L_b = \mathcal{L}|_{X_b}$, and assume $\dim_{\mathbb{C}} H^0(X_b, \mathcal{O}(L_b))$ is independent of $b \in B$.

b.) We assume we are given a singular toric action on X_1 with moment polytope Δ (such as a G-T system), and that the singular Hamiltonians have extensions to $X_b, b \in \mathcal{B}$ which converge towards the Hamiltonians of the toric variety X_0 as $b \to 0$. These systems on each X_b have moment polytope Δ, independent of $b \in \mathcal{B}$.

c.) We are vague about the convergence, which can be technical at this level. What we really want is simply: the *normalized* Bohr-Sommerfeld distributions $\tilde{s}_{\Lambda_{m(b)}}$(of *mass 1*) converge weakly to $\tilde{s}_{\Lambda_{m(0)}}$ as distributions, for all $m \in \Delta_{\mathbb{Z}}$.

Our main statement here is the following *perturbation result*:

Theorem 2. Given the set-up above, we have that the exact Bohr-Sommerfeld correspondence holds on X_b, for b close to 0.

Remark. Theorem 2 seems local in nature, but in all examples known to verify the set-up, any system on X_1 is, in fact, equivalent to a system on X_b for b arbitrarily close to 0, which would give that exact Bohr-Sommerfeld holds for X_1 as well.

Proof. From our result on singular toric varieties, we know $B(\tilde{s}_{\Lambda_m(0)}), m \in \Delta_{\mathbb{Z}}$ is a basis, and we know by the constancy of dimension that $R^0\pi_*(\mathcal{L})$ is locally free, with fibers $H^0(X_b, \mathcal{O}(L_b))$ for the corresponding vector bundle E. We can extend the basis of $H^0(X_0, \mathcal{O}(L_0))$ to a local holomorphic frame $\sigma_1, \ldots, \sigma_d$, of E, $d = \dim_{\mathbb{C}} H^0(X_0, \mathcal{O}(L_0)) = |\Delta_{\mathbb{Z}}|$. We note that the measures $\frac{1}{n!}\omega_b^n$ vary continuously on X_b's, so that we can orthonormalize the σ_i's continuously, and then the Bergman kernels vary continuously, too, in b. This follows from the construction in terms of the Gramm-Schmidt process. Finally consider the $d \times d$ complex matrix-valued function $\mathcal{M}(b)$, given by

$$\mathcal{M}(b) = (\mathcal{M}_{i,j}(b)) =$$

$$(\langle B_b(\tilde{s}_{\Lambda_m}), \sigma_j(b)\rangle) = \left(\left\langle \tilde{s}_{\Lambda_{m(b)}}, \sigma_j(b)\right\rangle\right),$$

where $m = m_i$ runs over $\Delta_{\mathbb{Z}}$, and $j = 1, \ldots, |\Delta_{\mathbb{Z}}|$. But this last is continuous in b and is invertible at $b = 0$. Thus exact Bohr-Sommerfeld holds for small b.

Remarks.

1.) For the Gelfand-Tsetlin system, this method does not yet give that the Bohr-Sommerfeld distributions are sent canonically to the so-called Gelfand-Tsetlin basis of H^0, since the singular torus action does not give an action on H^0. However, the original papers of Gelfand and Tsetlin showed these basis elements were the common eigenfunctions of an abelian algebra of quantum observables (operators) of higher order. It remains to see whether these operators have good limits as $b \to 0$, and how they operate on the Bohr-Sommerfeld distributions $\tilde{s}_{\Lambda(b)}$. The classical observables are the Hamiltonians, and they correspond to first order operators – their flows – on L. In the very simple example 1 above, the square z^2 is smooth and corresponds to the second order operator $(\zeta\frac{\partial}{\partial\zeta})^2$, which does behave well as $b \to 0$. The Bohr-Sommerfeld distributions are eigensections for this operator, for all $b \in \mathbb{C}$, as is seen in the last section.

2.) The above framework for showing the exact Bohr-Sommerfeld correspondence does not seem to have an immediate "physical interpretation." R. Szőke (1998) has studied (implicitly) this kind of degeneration in one case as a limit as $\hbar \to 0$.

3.) Recently Baier-Florentino-Mourão-Nunes have studied a large complex structure limit to retrieve the real polarization as a limit of the complex polarization on a toric manifold. It is not known whether one can take such a limit for singular toric systems such as Gelfand-Tsetlin, and deduce anything about the Bohr-Sommerfeld correspondence in this way.

4.) This degeneration method already appears for non-singular toric systems. Consider a Hirzebruch surface $F_{2k}, k \geq 1$, which is a toric variety. We can deform it smoothly to an $F_{2k'}$ for any $k' < k$. Under this deformation, we can extend the symplectic structure, the holomorphic line bundle and the Hamiltonian action. By Delzant's theorem, this system on $F_{2k'}$ has a holomorphic realization, which is just our original system on F_{2k}. This system cannot be represented holomorphically on $F_{2k'}$, so there is no way to use holomorphic toric equivariance to show exact Bohr-Sommerfeld for this system on $F_{2k'}$.

4. Relation to Geodesic Flow.

We have seen that for a Zoll manifold K the symplectic cut $T^*K//_{r=r_0}S^1 = M_{r_0}$ gave a compact model of the geodesic flow. In the case K is a compact symmetric space of rank 1 (CROSS), there is a natural complex structure on T^*K (Guillemin-Stenzel, Lempert-Szőke), which is $Isom^o(K)$-equivariantly biholomorphic to an affine variety (Patrizio-Wong). By rescaling, one can compactify this to a projective variety X. E.g., $K = S^n, T^*K = \mathcal{Q}^n_{aff}, X = \mathcal{Q}^n$. Note that $\mathcal{Q}^n \setminus \mathcal{Q}^n_{aff} = \mathcal{Q}^{n-1}$, which enables inductive procedures.

Definition. Given a (singular) completely integrable system $\mathcal{H}$ on $T^*_{r_0}K/S^1$, there is a (singular) CIS on T^*K, the *suspension* of $\mathcal{H}$, containing r as the new Hamiltonian. The moment polytope of this system is the suspension of the original one.

Example. If $K = S^3$, and we consider $\mathcal{Q}^2$ as a toric variety, we suspend the toric system to $T^*S^3//_{r=r_0}S^1$, where it gives the geodesic system as completely integrable, taken with a T^2 of isometries acting on S^3. The moment polytope is the suspension of a square (a "pyramid"), and the non-Delzant vertex is equivalent to the non-Delzant vertex in the Gelfand-Tsetlin polytope for $\mathbb{F}_3$. Note that anytime we take the suspension of any integrable system on $\mathcal{Q}^{n-1}$, we display the (compactified) geodesic flow on $\mathcal{Q}^n$ as completely integrable.

The *toric rank* of a singular toric system is the dimension of the subgroup which acts by holomorphic automorphisms. The toric rank of the last example is 2. Suspension preserves the toric rank.

There is one more inductive procedure called *push-out* from $\mathcal{Q}^{n-1}$ to $\mathcal{Q}^n$ which will raise the toric rank by 1. One cannot repeat push-out successively.

Theorem 3. One can use the above moves to show the geodesic system is completely integrable on S^n. The system can be taken to have maximal toric rank if n is odd.

Corollary. One can degenerate these systems on $\mathcal{Q}^n$ to toric varieties so that exact Bohr-Sommerfeld holds for them.