Z_2-index of the grassmanian G_{2n}^n

Roman Karasev
r_n_karasev@mail.ru

Roman Karasev, Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700

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For example, the Schubert calculus and other topological facts can be applied to obtain some existence theorems for flat transversals (affine flats intersecting all members of a given family of sets).
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This space has a natural Z_2-action (involution) by taking the orthogonal complement of the subspace.
The well-known invariant of Z_2-spaces is homological index, introduced and studied by Krasnosel'skii, Schwarz, Yang, Conner, and Floyd.

Let us state the theorem that gives an estimate for the index of the Grassmannian.

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Theorem

If $n = 2^l(2m+1)$, *then*

$$2^{l+1} - 1 \leq \operatorname{ind} G_{2n}^n \leq 2n - 1,$$

for $n = 2m+1$ *the index equals* 1, *for* $n = 2(2m+1)$ *the index equals* 3.

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for $n = 2m+1$ the index equals 1, for $n = 2(2m+1)$ the index equals 3.

The lower and the upper bounds coincide for $n = 2^l$, odd n, $n = 2(2m+1)$. In other cases there is still some gap between them.

Theorem 1 easily produces some geometric consequences. Here is one example (it also uses the Borsuk-Ulam theorem for Z_2-action).

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Corollary

Let $n = 2^l(2m+1)$, $k = 2^{l+1} - 1$. *Consider some* k *continuous (in the Hausdorff metric)* $O(n)$*-invariant functions* $\alpha_1, \dots, \alpha_k$ *on (convex) compacts in* $\mathbb{R}^n$. *Then for any (convex) compact* $K \subseteq \mathbb{R}^{2n}$ *there exist a pair of orthogonal* n*-dimensional subspaces* L *and* M*, such that for their respective orthogonal projections* π_L *and* π_M *we have*

$$\forall i = 1, \dots, k \; \alpha_i(\pi_L(K)) = \alpha_i(\pi_M(K)).$$

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Corollary

Let $n = 2^l(2m+1)$, $k = 2^{l+1} - 1$. Consider some k continuous (in the Hausdorff metric) $O(n)$-invariant functions $\alpha_1, \dots, \alpha_k$ on (convex) compacts in $\mathbb{R}^n$. Then for any (convex) compact $K \subseteq \mathbb{R}^{2n}$ there exist a pair of orthogonal n-dimensional subspaces L and M, such that for their respective orthogonal projections π_L and π_M we have

$$\forall i = 1, \dots, k \ \alpha_i(\pi_L(K)) = \alpha_i(\pi_M(K)).$$

In this corollary α_i can be the Steiner measures (volume, the boundary measure, the mean width, etc.) for example. The same statement holds if we consider a point $x \in K$ and sections of K by mutually orthogonal affine n-subspaces L and M through x, instead of projections to L and M.

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Denote $G = Z_2$, and note that

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Since any G-space X can be mapped to the point $\pi_X : X \to \mathrm{pt}$, we always have a natural map $\pi_X^* : \Lambda \to H_G^*(X, Z_2)$, the image c under this map will be denoted by c, if it does not make a confusion.

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Definition

The *cohomology index* of a Z_2-space X is the maximal n such that the power $c^n \neq 0$ in $H_G^*(X, Z_2)$. If there is no maximum, we consider the index equal to ∞. Denote the index of X by $\operatorname{ind} X$.

We also need the following well-known lemma (the generalized Borsuk-Ulam theorem for odd maps).

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Lemma

If there exists an equivariant map $f : X \to Y$*, then* $\operatorname{ind} X \leq \operatorname{ind} Y$*.*

The upper bound in Theorem 1 is given by the following lemma.

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Lemma

$$\operatorname{ind} G_{2n}^n \le 2n - 1.$$

Its proof uses an explicit construction of $2n$ odd functions (w.r.t. the considered Z_2-action) on G_{2n}^n, having no common zero.

We also have a lower bound for composite n.

Lemma

Suppose $n = ds$ for some positive integers d, s. Then

$$\text{ind}\, G_{2n}^n \geq \text{ind}\, G_{2d}^d.$$

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Lemma

Suppose $n = ds$ for some positive integers d, s. Then

$$\text{ind } G_{2n}^n \geq \text{ind } G_{2d}^d.$$

Its proof follows from existence of a Z_2-equivariant map $G_{2d}^d \to G_{2n}^n$.

In order to prove Theorem 1 it remains to prove the following lemmas.

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Lemma

If n is odd, then $\operatorname{ind} G_{2n}^n = 1$, *if* $n = 2 \mod 4$, *then* $\operatorname{ind} G_{2n}^n = 3$.

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Lemma

If $n = 2^l$, *then* $\operatorname{ind} G_{2n}^n = 2n - 1$.

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In order to prove the final lemmas, we have to describe the cohomology of the subgroup $G \subset O(2n)$, generated by the subgroup $O(n) \times O(n)$ (from some decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$), and Z_2 that interchanges the summands $\mathbb{R}^n$.

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$O(n) \wr Z_2 = (O(n) \times O(n)) \rtimes Z_2$.
In order to describe the cohomology (we always use cohomology modulo 2) of a wreath product, we have to use the construction of external Steenrod squares. This construction is known and was used in [Hung 1990, Theorem 2.1] to describe the modulo 2 cohomology of the symmetric group and configuration spaces.

Let us give some definitions and lemmas in the spirit of the book [Buoncristiano, Rourke, Sanderson 1976].

Definition

Define the mock bundle

$$\xi \odot \eta = (\xi \times \eta \times S^n + \eta \times \xi \times S^n)/Z_2,$$

where Z_2 exchanges the components $\xi \times \eta$ and $\eta \times \xi$.

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Lemma

Denote c the hyperplane class in $H^1(\mathbb{R}P^n)$. Then for any $\xi, \eta \in H^*(K)$ the product

$$(\xi \odot \eta) \smile \sigma_K^*(c) = 0$$

in $H^*((K \times K \times S^n)/Z_2)$.

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Definition

Let $\xi : E(\xi) \to K$, $\eta : E(\eta) \to K$ be two mock bundles. Let p_+, p_- be the north and the south poles of S^n. Denote the mock bundle over $(K \times K \times S^n)/Z_2$

$$\iota(\xi \times \eta) = (\xi \times \eta \times \{p_+\} + \eta \times \xi \times \{p_-\})/Z_2.$$

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$$\iota(\xi \times \eta) = (\xi \times \eta \times \{p_+\} + \eta \times \xi \times \{p_-\})/Z_2.$$

It is obvious from the definition that we have relation

$$\iota(\xi \times \eta) \smile \sigma_K^*(c) = 0,$$

it is also obvious that

$$\iota(\xi \times \xi) = \mathrm{Sq}_e \, \xi \smile \sigma_K^*(c)^n.$$

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$$(\xi \odot \eta) \smile (\zeta \odot \chi) = (\xi \smile \zeta) \odot (\eta \smile \chi) + (\xi \smile \chi) \odot (\eta \smile \zeta),$$

$$(\xi \odot \eta) \smile (\mathrm{Sq_e}\, \zeta) = (\xi \smile \zeta) \odot (\eta \smile \zeta),$$

$$(\xi \odot \eta) \smile \iota(\zeta \odot \chi) = \iota((\xi \smile \zeta) \times (\eta \smile \chi)) + \iota((\xi \smile \chi) \times (\eta \smile \zeta)),$$

$$\mathrm{Sq_e}\, \xi \smile \mathrm{Sq_e}\, \eta = \mathrm{Sq_e}(\xi \smile \eta),$$

$$\mathrm{Sq_e}\, \xi \smile \iota(\eta \times \zeta) = \iota((\xi \smile \eta) \times (\xi \smile \zeta)),$$

$$\iota(\xi \times \eta) \smile \iota(\zeta \times \chi) = 0.$$

Let us describe the structure of the cohomology $H^*((K \times K \times S^n)/Z_2)$ after making definitions.

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Definition

Consider a Z_2-algebra A with linear basis $v_1, \dots, v_n$. Denote $A \odot A$ the subalgebra of $A \otimes A$, invariant w.r.t. Z_2-action by permutation. The linear base of A is

$$\{v_i \otimes v_i\}_{i=1}^n, \ \{v_i \otimes v_j + v_j \otimes v_i\}_{i<j}.$$

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Definition

Consider a Z_2-algebra A with linear basis $v_1, \dots, v_n$. Denote $\iota(A \otimes A)$ the quotient vector space $A \otimes A/(v_i \otimes v_j + v_j \otimes v_i)$. As Z_2-algebra it has zero multiplication.

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Lemma

The maps Sq_e, $\odot$, *map the algebra* $H^*(K) \odot H^*(K)$ *to* $H^*((K \times K \times S^n)/Z_2)$. *The map* ι *maps* $\iota(H^*(K) \otimes H^*(K))$ *to* $H^*((K \times K \times S^n)/Z_2)$. *The images of these maps generate the cohomology* $H^*((K \times K \times S^n)/Z_2)$.
The latter cohomology can be described as the quotient of $H^*(K) \odot H^*(K) \otimes Z_2[c] \oplus \iota(H^*(K) \otimes H^*(K))$ *by the relations*

$$c^{n+1} = 0,\ (\xi \odot \eta) \otimes c = 0, \mathrm{Sq}_e\, \xi \otimes c^n = \iota(\xi \otimes \xi).$$

the c *is the preimage of the hyperplane class in* $H^1(\mathbb{R}P^n)$.

We also need to know the behavior of Stiefel-Whitney classes of vector bundles under the external Steenrod square operation, given by the following lemma.

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Lemma

Let $\dim \nu = k$*, and let the Stiefel-Whitney class of* ν *be*

$$w(\nu) = w_0 + w_1 + \ldots + w_k.$$

Then

$$w(\operatorname{Sq_e} \nu) = \sum_{0\le i<j\le k} w_i \odot w_j + \sum_{i=0}^{k} (1+c)^{k-i} \operatorname{Sq_e} w_i,$$

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Lemma

The kernel of the natural map $\pi^* : H^*(BG) \to H^*(O(2n)/G)$ is generated by the homogeneous components of positive degree of the expression

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The final lemmas follow from this description by straightforward calculations of the nilpotency degree of the element c under the above relations.

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The author thanks Oleg Musin for drawing attention to the problem of calculating the Z_2-index of G_{2n}^n and for the discussion of these results, and Peter Landweber for the discussion of the topological technique.

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