Finite group actions on aspherical spaces

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   - Homotopy classification
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Layout

Let p be an action of a finite group G on a topological space T.

Problem

Classify all actions of finite groups on topological spaces up to homotopy conjugation.

Certainly we can require some restrictions on space T.
For example we can assume that T or $T/p(G)$ is a CW-complex.

Let p_1 and p_2 be actions of finite groups G_1 and G_2 on topological spaces T_1 and T_2.

Homotopy conjugation

If there exist a homotopy equivalence $\varphi : T_1 \to T_2$ and an isomorphism $\theta : G_1 \to G_2$ such that

$$\varphi \circ p_1(g) = p_2\big(\theta(g)\big) \circ \varphi \text{ for all } g \in G_1,$$

then we say that the actions p_1 is homotopy conjugate to the action p_2 and write $p_1 \sim p_2$.

Example of $\sim$ asymmetry

Suppose $\mathbb{Z}_2 = \{\pm 1\}$ acts on an infinite dimensional sphere S^∞ in two ways: trivially $p_1(\pm 1)x = x$ and freely $p_2(\pm 1)x = \pm x$; then $p_2 \sim p_1$ by homotopy equivalence $\varphi : S^\infty \to S^\infty$ such that

$$\varphi : S^\infty \mapsto x_0 \in S^\infty.$$

But there is no map $\varphi' : S^\infty \to S^\infty$ such that

$$\varphi' \circ p_1(\pm 1) = p_2(\pm 1) \circ \varphi'.$$

So, $p_1 \nsim p_2$.

Regular free actions

A free action p of a finite group G on a space T is called regular if the space $T/p(G)$ is a CW-complex.

Clearly if p is a regular then the space T is also a CW-complex.

Lemma (A)

Suppose the actions p_1 and p_2 are free and regular, and the spaces T_1 and T_2 are aspherical; then $p_1 \sim p_2$ iff $p_2 \sim p_1$.

It is obvious that if $p_1 \sim p_2$ then spaces T_1 and T_2 homotopy equivalent.

Moreover using the Whitehead theorem we see that if T_1 and T_2 are aspherical then $T_1/p_1(G)$ and $T_2/p_2(G)$ are homotopy equivalent too.

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Let p be a free action of a finite group G on an aspherical space T of type $K(\pi, 1)$. Therefore the long exact sequence of the regular covering $P : T \to T/p(G)$ has the form

$$1 \to \pi \to \pi_1\big(T/p(G)\big) \to G \to 1.$$

Thus we have an extension of G by π. This extension is called a subordinate extension to the free action p.

Theorem (A)

Let G be a finite group and π a discrete group. Then the set of all regular free actions (up to homotopy conjugation) of the group G on aspherical spaces of type $K(\pi, 1)$ is in one-to-one correspondence with the set of all classes of equivalent extensions of G by π. Here an action p corresponds to the subordinate extension.

Classification of discrete group extnsions (up to congruence) is well-known (Eilenberg, MacLane, 1947).

Abelian fundamental group

If the group π is abelian then the action p of G on T induces the action η of G on π:

$$\eta : G \to \mathrm{Aut}\ \pi.$$

G-module structure on π is defined by the map η. The set of all congruence classes of extensions of G by π is exactly the second cohomology $H^2(G, \pi)$ (the structure of G-module on π is fixed).

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Extensions of $\mathbb{Z}_2$ by $\mathbb{Z}$

Suppose $G = \mathbb{Z}_2$ and $\pi = \mathbb{Z}$. Let $T = S^1 \times S^\infty$. Thus there exist only three different extensions of $\mathbb{Z}_2$ by $\mathbb{Z}$:

$$\begin{array}{llllll} 1 \to \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \times \mathbb{Z} & \to \mathbb{Z}_2 \to 1; & p_1(\pm 1)(x,y) = (x, \pm y) \\ 1 \to \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \to \mathbb{Z}_2 \to 1; & p_2(\pm 1)(x,y) = (\pm x, y) \\ 1 \to \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \ltimes \mathbb{Z} & \to \mathbb{Z}_2 \to 1; & p_3(\pm 1)(x,y) = (x^{\pm 1}, \pm y) \end{array}$$

Here $(x,y) \in T = S^1 \times S^\infty$ and $S^1 \subseteq \mathbb{C}$.

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Let p be an action (optionally non-free) of a finite group G on an aspherical space T.

Subordinate extension in non-free case

The diagonal free action p_f is induced on the space $T \times \mathbf{E}G$ where $\mathbf{E}G$ is a contractible space and G acts regularly and freely on it. The following extension is called a subordinate extension to the action p:

$$1 \to \pi_1(T) \to \pi_1(T \times \mathbf{E}G/G) \to G \to 1.$$

Lemma (B)

The notion of subordinate extension is well defined, i.e., if the action p is free, then the subordinate extensions in first and second sense are equivalent.

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There exists a Cartan-Serre's spectral sequence for the diagonal action of G on $T \times \mathbf{E}G$:

$$E_2^{p,q} = H^p\big(G, H^q(T)\big) \Rightarrow H^{p+q}\big((T \times \mathbf{E}G)/G\big).$$

Here the abelian groups $H^*(T)$ are G-modules induced by the action p. Since the space T is aspherical; then $H^*(T)$ and $H^*\big((T \times \mathbf{E}G)/G\big)$ are a group cohomology and the previous spectral sequence is the same as a spectral sequence of Hochschild-Mostov for the subordinate extension:

$$E_2^{p,q} = H^p\big(G, H^q(T)\big) \Rightarrow H^{p+q}\Big(\pi_1\big((T \times \mathbf{E}G)/G\big)\Big).$$

Thus there is an information about relations between cohomology of the group G and the space T in the subordinated extension.

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Lemma (C)

If the action p has a fixed point x_0, then the subordinate extension is decomposed, i.e., it is a semidirect product. Here the structure of the semidirect product on $\pi_1\big((T \times \mathbf{E}G)/G\big)$ is defined by the action of G on $\pi_1(T, x_0)$.

Example

Let $\mathbb{Z}_2$ act on a circle $S^1 \subseteq \mathbb{C}$ by reflection: $p(\pm 1)x = x^{\pm 1}$. Then $\mathbf{E}\mathbb{Z}_2 = S^\infty$ and $p_f = p_3$, i.e., $p_f(\pm 1)(x, y) = (x^{\pm 1}, \pm y)$ where $(x, y) \in S^1 \times S^\infty$. So the subordinate extension is

$$1 \to \mathbb{Z} \to \mathbb{Z}_2 \ltimes_{(-1)} \mathbb{Z} \to \mathbb{Z}_2 \to 1.$$

Lemma (D)

Let a finite group G act on an aspherical space T and let H be its subgroup. Consider the induced action of H on T. Then the following diagram is commutative and has exact rows and columns.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(T) & \longrightarrow & \pi_1\big((T \times \mathbf{E}H)/H\big) & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow id & & \downarrow P_{\#} & & \downarrow & & \\
1 & \longrightarrow & \pi_1(T) & \longrightarrow & \pi_1\big((T \times \mathbf{E}G)/G\big) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & \frac{\pi_1((T\times\mathbf{E}G)/G)}{\pi_1((T\times\mathbf{E}H)/H)} & \longrightarrow & G/H & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

Layout

Let P be a poset. Consider P as a small category in natural sense. By $\mathbf{B}P$ denote the classifying space of the small category P.

Let a finite group G act on P with respect to the order. Then this action naturally induces an action on the space $\mathbf{B}P$.

By $\mathrm{C}G$ and $\mathrm{L}G$ denote the coset poset and the subgroup poset of G respectively.

Theorem (B)

Let L be one of the posets $\mathrm{C}G$ or $\mathrm{L}G$ and let an action of G on L (by conjugation or shift) be fixed. If the classifying space $\mathbf{B}L$ is aspherical, then there exists a cohomological spectral sequence convergent to the cohomology of the group S:

$$E_2^{p,q} \;=\; H^p\big(G, H^q(L)\big) \;\Rightarrow\; H^{p+q}(S),$$

where $1 \to \pi_1(\mathbf{B}L) \to S \to G \to 1$ is the exension subordinated to the action of G on $\mathbf{B}L$.

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Since any connected graph is an aspherical space; then any action of a finite group on it induces a subordinate extension.

Dihedral group

Let the dihedral group $G = D_{2n}$, $n \geq 3$ act naturally on a circle $T = S^1$ and let $H = \mathbb{Z}_n \subseteq D_{2n}$. Then using lemma D we get

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \to & \mathbb{Z}_n & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbb{Z} & \to & A & \to & D_{2n} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Consequently, $A = \mathbb{Z}_2 \ltimes \mathbb{Z}$ and $1 \to \mathbb{Z} \to \mathbb{Z}_2 \ltimes \mathbb{Z} \to D_{2n} \to 1$ is the subordinate extension.

S. Eilenberg, S. MacLane , "Cohomology theory in abstract groups. II. Group extensions with a non-abelian kernel", Ann. Math., 1947, (2) 48, p. 199-236.

J. Shareshian, "On the shellability of the order complex of the subgroup lattice of a finite group", Trans. Amer. Math. Soc. 353 (2001), 2689–2703.