Cones of effective two-cycles on toric manifolds

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X: smooth projective algebraic variety over $\mathbb{C}$
(projective toric manifold)

$d := \dim X$

Definition 1 X *is Fano if* $-K_X = \mathrm{c}_1(X) = \mathrm{ch}_1(X)$ *is ample.*

$$\#\{\text{smooth toric Fano } d\text{-folds up to } \cong\} < +\infty$$

Therefore, we can consider the classification problem for toric Fano manifolds.

∘ (classical) Results

dimension	# of toric Fano	
1	1	$\mathbb{P}^1$
2	5	$\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, F_1, S_7, S_6
3	18	Batyrev, Watanabe-Watanabe
4	124	Batyrev, Sato

- Picard number $\rho(X) = 1, 2, 3, 2d-1, 2d$ (upper bound).

- Symmetric and pseudo-symmetric.

- With a divisorial contraction to a point or a curve.

- Certain fixed indices.

- $\cdots$

◦ (recent) Results

dimension	# of toric Fano	
5	866	Kreuzer-Nill
6	7622	Øbro
7	72256	Øbro
⋮	⋮	⋮

◦ Øbro (2007) constructed an algorithm SFP.

$d \in \mathbb{N}$ (input) $\longrightarrow$ SFP $\longrightarrow$ all the toric Fano d-folds (output)

$\Longrightarrow$ The classification of toric Fano manifolds has been completed.

Definition 2 (Starr) *A Fano manifold X is a 2-Fano manifold if $ch_2(X) = \frac{1}{2}(c_1^2 - 2c_2)$ is a nef 2-cocycle.*

Definition 3 *A 2-cocycle E is* nef *if $(E \cdot S) \geq 0$ for any surface S on X.*

Example 4 (1) $\mathbb{P}^d$.

(2) $X \times Y$ (X, Y 2-*Fano*).

(3) *complete intersection* $(d_1, \ldots, d_c) \subset \mathbb{P}^n$ *s.t.* $\sum d_i^2 \leq n + 1$.

(4) *Grassmannian* $G(k, n)$ *s.t.* $2k \leq n \leq 2k + 2$.

For more (non-toric) examples, see Araujo-Castravet (arXiv 0906.5388).

∘ Why 2-Fano?

Remark 5 *There are some results about the existence of rational surfaces on 2-Fano manifolds.*

Remark 6 *Without the assumption X: Fano, there exist infinitely many such toric manifolds of a fixed dimension (You will see later).*

Remark 7 *For the property $X \times Y$: 2-Fano (X, Y: 2-Fano), we need the assumption $ch_2(X)$: nef (not positive).*

Today Which toric Fano manifold is a 2-Fano manifold?

From now on, X: projective toric manifold .

Remark 8 *Of course, 2-Fano $\Rightarrow$ Fano by definition, only we have to do is the calculation of intersection numbers.*

However, I want to do this classification by using a Mori theoretical method.

2-Mori theory?

$$\begin{array}{ccc} \text{Mori theory} & \leftrightarrow & \text{rational curves} \\ \text{2-Mori theroy} & \leftrightarrow & \text{rational surfaces} \end{array}$$

◦ Preliminay

$X = X_\Sigma$: projective toric manifold associated to a fan Σ.

$$\begin{aligned} \mathrm{G}(\Sigma) &= \{\text{the primitive generaters of 1-dimensional cones in } \Sigma\} \\ &= \{x_1, x_2, \ldots\} \end{aligned}$$

$Z_2(X) := \{\sum a_i S_i\}$ the group of 2-cycles on X.

$Z^2(X) := \{\sum b_i E_i\}$ the group of 2-cocycles on X.

<u>intersection pairing</u>

$$\begin{aligned} Z^2(X) \times Z_2(X) &\longrightarrow \mathbb{Z} \\ (E, S) &\longmapsto (E \cdot S) \end{aligned}$$

numerical equivalence

$$S \equiv 0 \iff (E \cdot S) = 0 \text{ for any } E \in Z^2(X)$$
$$E \equiv 0 \iff (E \cdot S) = 0 \text{ for any } S \in Z_2(X)$$

$N_2(X) := (Z_2(X)/\equiv) \otimes \mathbb{R}$ and $N^2(X) := (Z^2(X)/\equiv) \otimes \mathbb{R}$

2-Mori cone

$$\begin{aligned} NE_2(X) &= \{\text{the numerical classes of effective 2-cycles}\} \\ &= \left\{ [\sum a_i S_i] \,\middle|\, a_i \geq 0 \right\} \subset N_2(X) \end{aligned}$$

Proposition 9 $NE_2(X) \subset N_2(X)$ *is a strongly convex polyhedral cone.*

So, $ch_2(X)$: nef $\Leftrightarrow$ $(ch_2(X) \cdot S) \geq 0$ for any extremal surface $S \in NE_2(X)$.

Remark 10 *In ordinary Mori theory,*
extremal ray $R \subset NE(X) \leftrightarrow$ *contraction* $\varphi_R : X \to \overline{X}$*.*
By this correspondence, we can find extremal curves easily.

Question For 2-Mori theory,
does there exist a correspondence like in this Remark?

◦ Simple example

- $\boxed{\mathbb{P}^1 \times \mathbb{P}^3}$ dim $N_2(X) = 2$.

$$NE_2(X) = \mathbb{R}_{\geq 0}[\text{pt.} \times \mathbb{P}^2] + \mathbb{R}_{\geq 0}[\mathbb{P}^1 \times \mathbb{P}^1].$$

- $\boxed{\mathbb{P}^2 \times \mathbb{P}^2}$ dim $N_2(X) = 3$.

$$NE_2(X) = \mathbb{R}_{\geq 0}[\text{pt.} \times \mathbb{P}^2] + \mathbb{R}_{\geq 0}[\mathbb{P}^1 \times \mathbb{P}^1] + \mathbb{R}_{\geq 0}[\mathbb{P}^2 \times \text{pt.}].$$

In each case, every extremal surface is contracted by a projection.

◦ An expression of numerical classes

$Y = Y_\sigma \subset X$: a T-invariant subvariety of $\dim Y = l$ associated to $\sigma \in \Sigma$.

$\mathrm{G}(\Sigma) = \{x_1, \ldots, x_m\}$. $x_i \leftrightarrow D_{x_i}$: T-invariant prime divisor.

$$I_Y = I_Y(X_1, \ldots, X_m) := \sum_{1 \leq i_1, \ldots, i_l \leq m} (D_{x_{i_1}} \cdots D_{x_{i_l}} \cdot Y) X_{i_1} \cdots X_{i_l}$$

$$\in \mathbb{Z}[X_1, \ldots, X_m] \quad (x_i \leftrightarrow X_i)$$

Remark 11 *I_Y has all the informations of intersection numbers of Y on X. So, we consider I_Y as the numerical class of $Y \in N_l(X)$.*

∘ Example

$C = C_\tau \subset X$: T-invariant curve.

$$I_C = \sum (D_i \cdot C) X_i$$

is a polynomial of degree 1.

$\Longrightarrow$ $\sum (D_i \cdot C) x_i = 0$ is a Reid's wall relation associated to τ.

Namely, I_C is calculated from the wall relation immediately.

$S = S_\tau \subset X$: be a T-invariant surface.

Theorem 12 *I_S is calculated as follows*:

(1) $\boxed{S \cong \mathbb{P}^2}$ *Let $C \subset S$ be a T-invariant curve. Then,*

$$I_S = (I_C)^2.$$

(2) $\boxed{S \cong F_\alpha}$ *Let $C_{\mathsf{f}} \subset S$ be a fiber of $S = F_\alpha \to \mathbb{P}^1$, while let C_{n} be the negative section of S. Then,*

$$I_S = \alpha (I_{C_{\mathsf{f}}})^2 + 2 I_{C_{\mathsf{f}}} I_{C_{\mathsf{n}}}.$$

(3) $\boxed{\textit{otherwise}}$ *Omit. Using the sequence*

$$F_\alpha \leftarrow \cdots\cdots \leftarrow S,$$

where $\leftarrow$ is a blow-up, we can calculate I_S explicitly.

∘ Intersection numbers with $ch_1(X)$ or $ch_2(X)$

(1)

$$-K_X = c_1(X) = ch_1(X) = X \setminus T = \sum_{x \in \mathsf{G}(\Sigma)} D_x$$

$$I_C = \sum_i a_i X_i \implies (ch_1(X) \cdot C) = \sum_i a_i$$

(2)

$$ch_2(X) = \frac{1}{2} \sum_{x \in \mathsf{G}(\Sigma)} D_x^2$$

$$I_S = \sum_{i,j} a_{ij} X_i X_j \implies (ch_2(X) \cdot S) = \frac{1}{2} \sum_i a_{ii}$$

◦ An application

X: toric manifold of $\rho(X) = 2$.

Theorem 13 *(Kleinschmidt) X is $\mathbb{P}^m$-bdl over $\mathbb{P}^n$.*

Let $X = X_\Sigma = \mathbb{P}_{\mathbb{P}^{n-1}}\left(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{m-1})\right)$,
where $a_1 \geq \cdots \geq a_{m-1} \geq 0$, $m + n - 2 = \dim X =: d$.

Put $\mathrm{G}(\Sigma) = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$, and let

$$x_1 + \cdots + x_m = 0, \tag{1}$$
$$y_1 + \cdots + y_n = a_1 x_1 + \cdots + a_{m-1} x_{m-1}, \tag{2}$$

be the extremal wall relations ($\leftrightarrow$ extremal ray of $NE(X)$).

$$(1) \leftrightarrow C_1$$
$$(2) \leftrightarrow C_2$$

First, we determine the extremal rays of $NE_2(X)$.

By calculating rational functions for a $\mathbb{Z}$-basis $\{x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1}\}$, we have the relations

$$
\begin{aligned}
D_1 - D_m + a_1 E_n &= 0, \\
D_2 - D_m + a_2 E_n &= 0, \\
&\cdots \\
D_2 - D_m + a_2 E_n &= 0, \\
E_1 - E_n &= 0, \\
E_2 - E_n &= 0, \\
&\cdots \\
E_{n-1} - E_n &= 0,
\end{aligned}
$$

where $D_1, \ldots, D_m, E_1, \ldots, E_n$ are
T-invariant prime divisors corresponding to $x_1, \ldots, x_m, y_1, \ldots, y_n$.

Therefore, for $1 \le i, j \le m-1$,

$$D_j = D_i + (a_i - a_j)E_n, \tag{3}$$

and

$$E_1 = E_2 = \cdots = E_n. \tag{4}$$

Every $(d-2)$-dimensional cone $\tau \in \Sigma$ is expressed as

$$\tau = \mathbb{R}_{\ge 0}\, x_{i_1} + \cdots + \mathbb{R}_{\ge 0}\, x_{i_k} + \mathbb{R}_{\ge 0}\, y_{j_1} + \cdots + \mathbb{R}_{\ge 0}\, y_{j_l}$$

for some $1 \le i_1 < \cdots < i_k \le m$, $1 \le j_1 < \cdots < j_l \le n$
s.t. $k < m$, $l < n$ and $k + l = d - 2$.

So, the corresponding T-invariant surface S_τ is

$$S_\tau = D_{i_1} \cdots D_{i_k} E_{j_1} \cdots E_{j_l} \in N_2(X).$$

By using (3) and (4),
any S_τ is expressed as a linear combination of 2-cycles

$$D_1 \cdots D_p E^q \qquad (p \leq m-1,\ q \leq n-1,\ p+q = d-2)$$

whose coefficients are non-negative (remark that $i < j \Rightarrow a_i - a_j \geq 0$).

Moreover, since $D_1 \cdots D_m = E_1 \cdots E_n = 0$ by wall relations (1) and (2), the possibilities for the generaters of $NE_2(X)$ are

$$S_1 := D_1 \cdots D_{m-3} E^{n-1},$$

$$S_2 := D_1 \cdots D_{m-2} E^{n-2} \text{ and}$$

$$S_3 := D_1 \cdots D_{m-1} E^{n-3}.$$

In fact, the following holds:

$$
\begin{aligned}
NE_2(X) &= \mathbb{R}_{\geq 0}\, S_1 + \mathbb{R}_{\geq 0}\, S_2 + \mathbb{R}_{\geq 0}\, S_3 && \text{if } m \geq 3,\ n \geq 3. \\
NE_2(X) &= \mathbb{R}_{\geq 0}\, S_2 + \mathbb{R}_{\geq 0}\, S_3 && \text{if } m = 2,\ n \geq 3. \\
NE_2(X) &= \mathbb{R}_{\geq 0}\, S_1 + \mathbb{R}_{\geq 0}\, S_2 && \text{if } m \geq 3,\ n = 2.
\end{aligned}
$$

On the other hand, for each case,
$\dim N_2(X) = 3$, $\dim N_2(X) = 2$ and $\dim N_2(X) = 2$, respectively.

So, $NE_2(X)$ is a simplicial cone for each case,
and S_1, S_2 and S_3 are extremal surfaces.

Next, we will check when X becomes a 2-Fano manifold.

○ Positivity of $ch_1(X)$

Let C_2 be the T-invariant curve which generates the extremal ray corresponding to the wall relation (2). Then,

$$(-K_X \cdot C_2) = n - (a_1 + \cdots + a_{m-1})\,.$$

Therefore, X is a Fano manifold if and only if

$$n - (a_1 + \cdots + a_{m-1}) > 0. \tag{5}$$

∘ Non-negativity of $ch_2(X)$

Since $S_1 \cong S_3 \cong \mathbb{P}^2$,
$(\mathrm{ch}_2(X) \cdot S_1) \geq 0$ and $(\mathrm{ch}_2(X) \cdot S_3) \geq 0$ are trivial by Theorem 12.

On the other hand, we can easily check that $S_2 \cong F_{a_{m-1}}$.
Again by Theorem 12, we have

$$I_{S_2} = a_{m-1}(I_{C_1})^2 + 2 I_{C_1} I_{C_2}$$

$$= a_{m-1} (X_1 + \cdots + X_m)^2 +$$

$$2 (X_1 + \cdots + X_m) (Y_1 + \cdots + Y_n - (a_1 X_1 + \cdots + a_{m-1} X_{m-1}))\,.$$

So, we obtain

$$(ch_2(X) \cdot S_2) = m a_{m-1} - 2(a_1 + \cdots + a_{m-1}). \tag{6}$$

In (6), suppose that $m \geq 3$ and $(ch_2(X) \cdot S_2) \geq 0$. Then,

$$(\mathrm{ch}_2(X) \cdot S_2) = (m-2)a_{m-1} - 2(a_1 + \cdots + a_{m-2}).$$

The assumption $a_1 \geq \cdots \geq a_{m-1} \geq 0$ says that

$$a_1 = \cdots = a_{m-1} = 0,$$

that is, $X \cong \mathbb{P}^{m-1} \times \mathbb{P}^{n-1}$.

On the other hand, suppose that $m = 2$ in (6).
Then, $(ch_2(X) \cdot S_2) = 0$, that is, $ch_2(X)$ is nef.

By (5), we can summarize as follows:

Theorem 14 *If X is a toric 2-Fano manifold of Picard number 2, then X is one of the following*:

(1) *A direct product of projective spaces.*

(2) $\mathbb{P}_{\mathbb{P}^{d-1}}(\mathcal{O} \oplus \mathcal{O}(a))$ $(1 \leq a \leq d-1)$.

Remark 15 *This calculation shows that there exist infinitely many projective toric manifolds of dimension d whose 2nd Chern character is nef ($\mathbb{P}^1$-bundles over $\mathbb{P}^{d-1}$).*

∘ The classification of toric 2-Fano manifolds of dimension at most 4.

dimension	# of toric Fano	# of toric 2-Fano
1	1	1
2	5	3
3	18	8
4	124	25

Remark 16 *Every toric 2-Fano manifold in this table is a direct product of other lower-dimensional toric 2-Fano manifolds or a* $\mathbb{P}^1$*-bundle over a lower-dimensional toric 2-Fano manifold.*