Torus fibrations and localization of index

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Joint work in progress with Hajime Fujita and Mikio Furuta

1. H. Fujita, M. Furuta, T. Y, *Torus fibrations and localization of index I*, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1-26.
2. H. Fujita, M. Furuta, T. Y, *Torus fibrations and localization of index II*, arXiv:0910.0358.
3. H. Fujita, M. Furuta, T. Y, *Torus fibrations and localization of index III*, coming soon.

Riemann-Roch number

(L, ∇^L) prequantum line bundle $\overset{\text{def}}{\Leftrightarrow}$ $\begin{cases} L\colon \text{Hermitian line bundle} \\ \nabla^L\colon \text{connection with } \frac{\sqrt{-1}}{2\pi} F_\nabla = \omega \end{cases}$

$\downarrow$

(M, ω) closed symplectic manifold

- Fix a compatible almost complex structure J

$$\Rightarrow \text{ Spin}^c \text{ Dirac operator } D\colon \Gamma(\bigwedge^\bullet T^*M^{0,1} \otimes L) \to \Gamma(\bigwedge^\bullet T^*M^{0,1} \otimes L)$$

- If (M, ω, J) is Kähler and L is holomorphic, then $D = \sqrt{2}(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L)$.

Definition (Riemann-Roch number)

$$RR(M, \omega) = \operatorname{ind} D = \dim \ker D^0 - \dim \ker D^1 \in \mathbb{Z}$$

- $RR(M, \omega)$ does not depend on the choice of J. Moreover,

$$RR(M, \omega) = \int_M e^\omega Td(M).$$

- If (M, ω, J) is Kähler and L is holomorphic, then

$$RR(M, \omega) = \sum_i (-1)^i \dim H^i(M, \mathcal{O}_L).$$

Riemann-Roch number

Example

$$(L, \nabla^L) = \left(\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy\right) / (x, y, z) \sim (x + n, y, e^{2\pi\sqrt{-1}ny} z)$$

$$\downarrow$$

$$(M, \omega) = \left((\mathbb{R}/\mathbb{Z})^2, dx \wedge dy\right)$$

- $RR(M, \omega) = \int_M e^{\omega} Td(M) = \int_M \omega = 1$

Bohr-Sommerfeld fiber

$$\pi : (M^{2n}, \omega) \to B^n \text{ Lagrangian fibration} \overset{\text{def}}{\Leftrightarrow} \begin{cases} \pi : \text{fiber bundle} \\ \omega|_{\text{fiber}} \equiv 0 \\ \dim \text{ fiber} = \frac{1}{2} \dim M \end{cases}$$

- $\forall$fiber $\cong \mathbb{R}^n/\mathbb{Z}^n$ ($\because$ Arnold-Liouville theorem)
- $(L, \nabla)|_{\text{fiber}}$ is a flat bundle.

Definition (Bohr-Sommerfeld (BS) fiber)

$\pi^{-1}(b)$ ($b \in B$) is said to be *Bohr-Sommerfeld* if $(L, \nabla)|_{\pi^{-1}(b)}$ is trivially flat.

- $\pi^{-1}(b)$ is BS $\Leftrightarrow$ $\exists$non-zero parallel section of $(L, \nabla)|_{\pi^{-1}(b)}$.
- BS fibers appear discretely.

Bohr-Sommerfeld fiber

Example (continued)

$$(L, \nabla^L) = \left(\mathbb{R} \times \mathbb{R}/\mathbb{Z} \times \mathbb{C}, d - 2\pi\sqrt{-1}xdy\right) / (x, y, z) \sim (x + n, y, e^{2\pi\sqrt{-1}ny} z)$$
$$\downarrow$$
$$(M, \omega) = \left((\mathbb{R}/\mathbb{Z})^2, dx \wedge dy\right)$$
$$\downarrow \pi(x, y) = x$$
$$B = \mathbb{R}/\mathbb{Z}$$

- $\pi^{-1}(x)$ is BS $\Leftrightarrow x = 0 \in \mathbb{R}/\mathbb{Z}$

  $\because$ For $s \in \Gamma(L|_{\pi^{-1}(x)})$ solving the equation

$$\begin{aligned} 0 &= \nabla^L_{\partial_y} s \\ &= \partial_y s - 2\pi\sqrt{-1}xs \end{aligned}$$
$$\therefore s = s(0)e^{2\pi\sqrt{-1}xy} \text{ (local solution)}$$

Since $\pi^{-1}(x) = \mathbb{R}/\mathbb{Z}$, s is global $\Leftrightarrow s(0) = s(1) = s(0)e^{2\pi\sqrt{-1}x} \Leftrightarrow$ $x = 0 \in \mathbb{R}/\mathbb{Z}$

RR=# BS

Theorem (Andersen '97)

$$RR(M, \omega) = \#BS \textit{ fibers}$$

- $RR(M, \omega)$ and #BS fibers correspond to the dimensions of the quantum Hilbert spaces of Spin^c quantization and the geometric quantization using a real polarization, respectively.

RR=# BS

Similar phenomena have been observed for Lagrangian fibrations “with singular fibers" and “degenerate" symplectic cases, such as,

- moment map of a nonsingular toric variety (Danilov '78)

$$RR(M,\omega) = \dim H^0(M;\mathcal{O}_L) = \#\mu(M)\cap\mathbb{Z}^n = \#\text{BS fibers}$$

- Gelfand-Cetlin's completely integrable system on the complex flag manifold (Guillemin-Sternberg '83)
- Goldman's completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemann surface (Jeffrey-Weitsman '92)
- Pre-symplectic toric manifolds (Karshon-Tolman '93)
- Torus manifolds (Masuda '99, Hattori-Masuda '03)

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These phenomena suggest a localization of the index to BS fibers.

Question

Make clear the mechanism that controls these phenomena.

Purpose

Purpose of this talk

To give a partial answer of this question. Namely,

1. *Define an “index" of a Dirac-type operator on an open manifold with certain geometric structure on the end.*
2. *A localization for the index.*
3. *Application to geometric quantization*

Main Theorem

W $\mathbb{Z}_2$-graded $Cl(TM)$-module bundle
$\downarrow$
M Riemannian manifold (possibly non-compact)
$\cup$
V open set

Theorem (Fujita-Furuta-Y '09)

Suppose that $M \setminus V$ is compact and V is equipped with an "acyclic compatible system $\{(\pi_\alpha, D_\alpha)\}_{\alpha \in A}$". Then, there exists an integer $\mathrm{ind}(M, V)$ *depending on all the data such that* $\mathrm{ind}(M, V)$ *satisfies the following properties.*

1. $\mathrm{ind}(M, V)$ *is invariant under continuous deformation of the data.*
2. *For M closed,* $\mathrm{ind}(M, V) = \mathrm{ind}\, D$.
3. *For $V = M$,* $\mathrm{ind}(M, V) = 0$ *(vanishing)*
4. *For $M' \subset M$ admissible open neighborhood of $M \setminus V$,*
$$\mathrm{ind}(M, V) = \mathrm{ind}(M', M' \cap V) \quad \textit{(excision)}$$
5. $\mathrm{ind}(M_1 \sqcup M_2, V) = \mathrm{ind}(M_1, M_1 \cap V) + \mathrm{ind}(M_2, M_2 \cap V)$ *(sum formula)*
6. *product formula "*$\mathrm{ind}((M_1, V_1) \times (M_2, V_2)) = \mathrm{ind}(M_1, V_1)\, \mathrm{ind}(M_2, V_2)$*"*

Main Theorem

Corollary (Localization)

Under the above assumption, suppose M is closed and there exists an open covering $M = \cup_{i=1}^{k} O_i \cup V$ such that $\{O_i\}$ are mutually disjoint. Then,

$$\operatorname{ind} D = \sum_{i=1}^{k} \operatorname{ind}(O_i, O_i \cap V)$$

What is an acyclic compatible system?
Case 1: $\mathbb{C}P^1$

Let $\mu\colon (M,\omega) = (\mathbb{C}P^1, 2\omega_{FS}) \to [0,2]$ be a moment map defined by

$$\mu([z_0 : z_1]) = 2\frac{|z_1|^2}{\|z\|^2}$$

and $(L,\nabla) = (H,\nabla)^{\otimes 2}$, where (H,∇) is the hyperplane bundle with connection.

- $W := \wedge^\bullet T^*M^{0,1} \otimes L = \wedge^{\text{even}} T^*M^{0,1} \otimes L \oplus \wedge^{\text{odd}} T^*M^{0,1} \otimes L$

  Note: $T^*\mu^{-1}(b) \otimes \mathbb{C} \cong T^*M^{0,1}|_{\pi^{-1}(b)}\ \forall b \in (0,2)$
- $D = \sqrt{2}(\bar\partial \otimes L + \bar\partial^* \otimes L)\colon \Gamma(W) \circlearrowleft$
- $\mu^{-1}(b)$ is BS $\Leftrightarrow b \in [0,2] \cap \mathbb{Z}$
- $V := M \setminus \mu^{-1}(\mathbb{Z})$
- O_i: an open neighborhood of a Bohr-Sommerfeld fiber

What is an acyclic compatible system?
Case 1: $\mathbb{C}P^1$

$$
\begin{aligned}
b \in \operatorname{Im}\mu \cap \mathbb{Z} &\Leftrightarrow \exists \text{parallel section } (\neq 0) \text{ of } (L,\nabla)|_{\mu^{-1}(b)} \\
&\Leftrightarrow \mathrm{H}^0(\mu^{-1}(b); (L,\nabla)|_{\mu^{-1}(b)}) \neq 0 \text{ (Note: } (L,\nabla)|_{\mu^{-1}(b)} \text{ flat)} \\
&\Leftrightarrow \mathrm{H}^\bullet(\mu^{-1}(b); (L,\nabla)|_{\mu^{-1}(b)}) \neq 0 \ (\because\ \mu^{-1}(b) : \text{torus}) \\
&\Leftrightarrow \text{The kernel of the de Rham operator } D_b \text{ of } \mu^{-1}(b) \text{ with} \\
&\qquad \text{coefficients in } L|_{\mu^{-1}(b)} \text{ is nontrivial.}
\end{aligned}
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$\Leftrightarrow$ The kernel of the de Rham operator D_b of $\mu^{-1}(b)$ with coefficients in $L|_{\mu^{-1}(b)}$ is nontrivial.

By bundling D_b w. r. t. b, we can obtain the following structure on V.

Acyclic compatible system -simplest case

- $\mu|_V : V \to \mu(V)$ S^1*-bundle*
- $D_{fiber} : \Gamma(\wedge^\bullet T^*[\mu|_V] \otimes L|_V) \circlearrowleft$ *de Rham operator along fibers of* $\mu|_V$.
- $D_{fiber} \circ c(\tilde{u}) + c(\tilde{u}) \circ D_{fiber} = 0 \ \forall b \in \mu(V) \ \forall u \in T_b\mu(V)$
- $\ker(D_{fiber}|_{\mu^{-1}(b)}) = 0 \ \forall b \in \mu(V)$

This is a simplest example of an acyclic compatible system.

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What is an acyclic compatible system?
Case 2: $\mathbb{C}P^1 \times \mathbb{C}P^1$

Let $(M, \omega) = (\mathbb{C}P^1, 2\omega_{FS}) \times (\mathbb{C}P^1, 2\omega_{FS})$ and $(L, \nabla) = (H, \nabla)^{\otimes 2} \boxtimes (H, \nabla)^{\otimes 2}$. Let us consider $\mu \times \mu$.

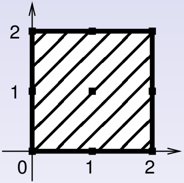


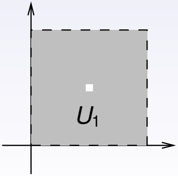
Figure: image of $\mu \times \mu$

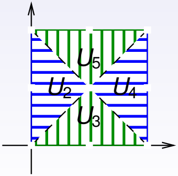
Put $V = M \setminus (\mu \times \mu)^{-1}(\mathbb{Z}^2)$. In this case $\mu|_V$ is no more T^2-bundle. But, locally there exist acyclic compatible systems on V, i.e.

What is an acyclic compatible system?

Structure on $V_i = \pi^{-1}(U_i)$ $(i = 1, \ldots, 5)$

- $$\begin{array}{ccccccccc} V = V_1 & \cup & V_2 & \cup & V_3 & \cup & V_4 & \cup & V_5 \\ \downarrow \pi_1 & & \downarrow \pi_2 & & \downarrow \pi_3 & & \downarrow \pi_4 & & \downarrow \pi_5 \\ V_1/S^1_1 \times S^1_2 & & V_2/S^1_2 & & V_3/S^1_1 & & V_4/S^1_2 & & V_5/S^1_1 \end{array}$$
- $\forall i$, $D_i : \Gamma\left(\wedge^\bullet T^*[\pi_i|_{V_i}] \otimes L|_{V_i}\right) \circlearrowleft$ *acyclic compatible system in the above sense.*
- *On each overlap, (π_i, D_i)'s satisfy some compatibility conditions.*
  Ex.
  $$D_i \circ D_j + D_j \circ D_i \text{ is non-negative on } \Gamma(W|_{V_i \cap V_j}).$$





Idea of proof
- Witten's deformation

$$\begin{array}{ll} (W, c) & \mathbb{Z}_2\text{-graded } Cl(TM)\text{-module bundle} \\ \downarrow & \\ M & \text{complete Riemannian manifold} \end{array}$$

For $t \geq 0$ define

$$D_t := D + th,$$

where $h \in \mathrm{End}(W)$ satisfying

- Hermitian
- degree-one
- $\mathrm{supp} h := \{x \in M \mid \ker(h_x \colon W_x \to W_x) \neq 0\}$ is compact
- $h \circ c + c \circ h = 0$

Point

ind D_t *is defined independently of* $\forall t \gg 0$ *in an appropriate sense. In particular,* ind D_t *is described in terms of the data restricted to a neighborhood of* supp$h$.

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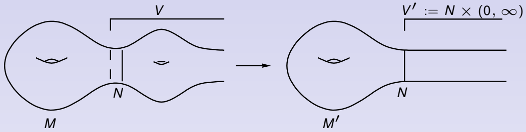
Outline of proof

1. Deform V cylindrically.



2. For $t \geq 0$ define

$$D_t := D + t \sum_{\alpha \in A} \rho_\alpha D_\alpha \rho_\alpha .$$

Fact (local index)

1. $\dim \ker D_t \cap L^2 < +\infty$ $(\forall t \gg 0)$.
2. *Moreover,* $\dim \ker D_t^0 \cap L^2 - \dim \ker D_t^1 \cap L^2$ *is independent of* $\forall t \gg 0$.

$$\operatorname{ind}(M, V) := \dim \ker D_t^0 \cap L^2 - \dim \ker D_t^1 \cap L^2 \in \mathbb{Z} \quad (\forall t \gg 0)$$

3. Check $\operatorname{ind}(M, V)$ is independent of a choice of a cut locus.

- A general Fredholm theory is necessary to prove the product formula.
- Similar to considering an “adiabatic limit"

Application to Lagrangian fibrations

Theorem (Fujita-Furuta-Y '08, '09)

For a prequantized closed Lagrangian fibration possibly with singular fibers, $RR(M, \omega)$ is localized at singular fibers and BS fibers.

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For a prequantized closed Lagrangian fibration without singular fibers,

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Theorem (Fujita-Furuta-Y '09)

For a prequantized four-dimensional closed locally toric Lagrangian fibration,

$$RR(M, \omega) = \#\text{(both singular and nonsingular) BS-fibers}.$$