Discrete and continuous complexes and posets in topological combinatorics

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Geometry, Topology,
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- posets
- (simplicial, polyhedral) complexes
- polytopes
- matroids

Continuous (topological) posets

Grassmannian posets $\mathcal{G}_n(\mathbb{R})$
configuration posets $exp_n(X)$

$$\Delta(\widetilde{\mathcal{G}}_n(\mathbb{R})) \cong S^{\binom{n}{2}-n+2}$$

$$\Delta(exp_n(S^1)) \cong S^{2n-1}$$

By definition, $\Delta(\mathcal{P})$ is the topological order complex, a.k.a. the “flag join” of $\mathcal{P}$, where $\Delta(\mathcal{P}) \subset \mathcal{P}_1 * \ldots * \mathcal{P}_n$ and

$$\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \in \Delta(\mathcal{P}) \Leftrightarrow x_1 < x_2 < \ldots < x_n.$$

- V.A. Vassiliev. Geometric realization of the homology of classical Lie groups and complexes, S–dual to flag manifolds. *St.–Petersburg Math. J.* 3:4, 108–115, 1991.
- V.A. Vassiliev. Continuous order complexes of subspaces, configurations and ideals, *Abstracts of lectures at the workshop "Geometric Combinatorics"*, MSRI, February 1997.
- V. Vassiliev, Topological order complexes and resolutions of discrimant sets, *Publ. Inst. Math. (Belgrade)* 66/80 (1999), 165-185.

Theorem: Let $I \subset \widetilde{\mathcal{G}}_n(R)$ be a closed ideal in the truncated Grassmannian poset $\widetilde{\mathcal{G}}_n(R)$ and let $\chi(I) = (\chi_1(I), \chi_2(I), \ldots)$ be the associated χ-vector. Then,

$$\chi(\Delta(I)) = \chi_1(I) + \chi_2(I) - \chi_3(I) - \chi_4(I) + \ldots$$
$$+ \ldots + (\sqrt{-1})^{n^2+n+2}\chi_n(I) + \ldots$$

Answers a question from, G-C. Rota. Ten Mathematics Problems I will never solve. *DMV mitteilungen* 2, 45–52, (1998).

Vassiliev "Geometric resolutions of singularities"

- Σ is a singular space, e.g. the space of singular matrices, polynomials with multiple zeros, singular knots, etc.
- $\Phi : \Sigma \longrightarrow \mathcal{P}$ is a (semi-continuous) map, where
- $\mathcal{P}$ is an associated topological poset of singularities.

The associated resolution is

$$\widetilde{\Sigma} := \bigcup_{x \in \Sigma} \{x\} \times \Delta(\mathcal{P}_{\geq \Phi(x)}) \subset \Sigma \times \Delta(\mathcal{P})$$

- G.M. Ziegler and R.T. Živaljević. Homotopy types of subspace arrangements via diagrams of spaces. *Math. Ann.*, 295:527–548, 1993.
- V. Welker, G.M. Ziegler, and R.T. Živaljević, Homotopy colimits; Comparison lemmas for combinatorial applications, *J. Reine Angew. Math.* 509 (1999), pp. 117-149.
- R.T. Živaljević, Combinatorics of topological posets. *Advances in Applied Mathematics*, Vol. 21 , Issue 4 (November 1998), 547 - 574.

$$\Delta(\widetilde{\Pi}_n) \simeq \bigvee_{i=2}^{n} \Sigma(\Delta(\widetilde{\Pi}^i_{n-1}))$$

$$\Delta(\widetilde{\mathcal{G}}_n(\mathbb{R})) \simeq S^{n-1} \wedge \Sigma(\Delta(\widetilde{\mathcal{G}}_{n-1}(\mathbb{R})))$$

$$\Delta(\widetilde{\mathcal{G}}^{\pm}_n(\mathbb{R})) \simeq (S^{n-1} \vee S^{n-1}) \wedge \Sigma(\Delta(\widetilde{\mathcal{G}}^{\pm}_{n-1}(\mathbb{R})))$$

$$\Delta(\exp_n(S^1)) \simeq S^n \wedge (\Delta(\widetilde{\mathcal{B}}_{n-1})/\partial\Delta(\widetilde{\mathcal{B}}_{n-1}))$$

$$\Delta(\mathcal{P}_n) \simeq P_n \wedge \Sigma(\Delta(\mathcal{P}_{n-1}))$$

$$\Delta(\exp_n(X)) \simeq \mathrm{Thom}_n(X \setminus \{x_0\})$$

Conjecture (V. Vassiliev),

“Geometric Combinatorics”, MSRI Berkeley, February 1997.

$$\Delta(exp_n(X)) \simeq X * \ldots * X \cong X^{*n}$$

By definition, $\mathrm{Thom}_n(Y)$ is the one-point-compactification of the space (vector bundle) $F(Y,n) \times_{S_n} \mathbb{R}^{n-1}$, hence

$$\Delta(exp_n(X)) \simeq \{F(X \setminus \{x_0\}, n) \times_{S_n} \mathbb{R}^{n-1}\} \cup \{+\infty\}.$$

As a consequence,

$$\Delta(exp_n(S^2)) \neq (S^2)^{*n}.$$

Continuous (topological) complexes

Weighted barycenters $\mathcal{B}_n(X)$

$\Omega(Q;G)$, the complex of vertex-colored polytopes

$\mathcal{B}_n(X) =$ The space of probability measarues with finite support $\subset X$.

Sadok Kallel, Rym Karoui, Symmetric Joins and Weighted Barycenters, arXiv:math/0602283v3 [math.AT] (June 2010).

$$\mathcal{B}_n(X)$$

- The union of all $(n-1)$-simplexes spanned by points in X,
- The ”space of barycenters” in non-linear analysis (A. Bahri and J.M. Coron); existence results for the solutions of nonlinear elliptic equations,
- The ”space of chords” in differential geometry.

Theorem (S. Kallel, R. Karoui)

$$\mathcal{B}_n(X) \simeq Sym^{*n}(X) := X^{*n}/S_n \text{ (symmetric join)}$$

$$\mathcal{B}_n(X) \simeq S^{n-1} \wedge_{S_n} X^{(n)}$$

and as a consequence

$$\mathcal{B}_2(S^n) \simeq \Sigma^{n+1}(\mathbb{R}P^n),$$

(I. James, E. Thomas, H. Toda, J.H.C. Whitehead).

$$\mathcal{B}_n(X) \cong \Delta(exp_n(X))!$$

Example: $exp_2(S^n)$ is a one-point compactification of

$F(\mathbb{R}^n, 2)\times_{\mathbb{Z}/2}\mathbb{R}^1 \cong \mathbb{R}^n\times(\mathbb{R}^n\backslash\{0\})\times_{\mathbb{Z}/2}\mathbb{R}^1 \cong \mathbb{R}^n\times\mathbb{R}^+\times(S^{n-1}\times_{\mathbb{Z}/2}\mathbb{R}^1)$

which immediately implies

$$\mathcal{B}_n(X) \cong \Delta(exp_n(X)) \simeq \Sigma^{n+1}(\mathbb{R}P^n).$$

Similarly,

$$\Delta(exp_n(X)) \simeq S^{n-1} \wedge_{S_n} X^{(n)}$$

which recovers the result of Kallel and Karoui.

Splitting a necklace; the case of k thieves

Theorem: (N. Alon, 1987) Let $\mu_1, \mu_2, \ldots, \mu_n$ be a collection of n continuous probability measures on $[0, 1]$. Let $k \geq 2$ and $N := n(k-1)$. Then there exists a partition of $[0, 1]$ by N cut points into $N+1$ intervals $I_0, I_1, \ldots, I_N$ and a function $f : \{0, 1, \ldots, N\} \to \{1, \ldots, k\}$ such that for each μ_i and each $j \in \{1, 2, \ldots, k\}$,

$$\sum_{f(p)=j} \mu_i(I_p) = 1/k.$$

Problem: What are higher dimensional analogues of Alon's splitting necklace theorem?

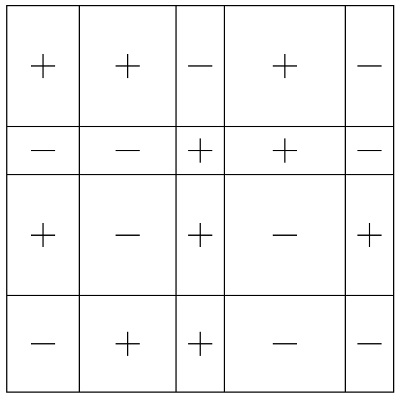

Figure 1: Splitting a square.

Higher dimensional necklace theorem

Theorem: (M. de Longueville, R. Ž., Adv. Math. 2008) Assume $n, d \geq 1$ and $k \geq 2$, and let $\mu_1, \ldots, \mu_n$ be a collection of n continuous probability measures on the d-dimensional cube $I^d \subset \mathbb{R}^d$. Then there exists a fair division with m_i hyperplane cuts parallel to i-th coordinate hyperplane if and only if

$$m_1 + \cdots + m_d \geq n(k-1). \quad (1)$$

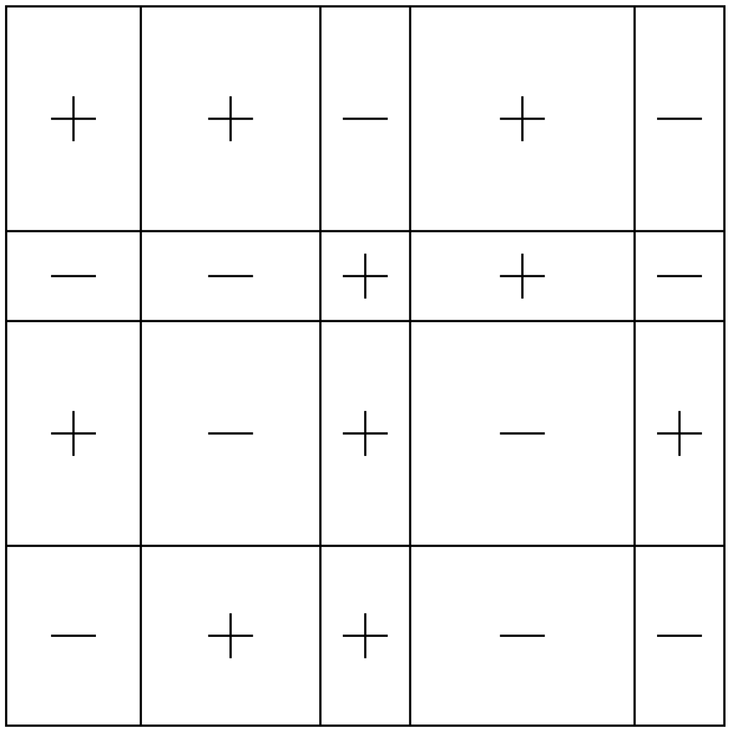

Figure 2: Splitting a square.

The complex $\Omega(Q;G)$ of vertex-colored polytopes

Relatives of $\Omega(Q;G)$ are

1. Inflated simplicial complexes

   (A. Björner, M.L. Wachs and V. Welker, Poset Fiber Theorems, Trans. AMS , 357 (2005), 1877–1899),

2. "Polyhedral join complexes" (relatives of generalized moment-angle complexes),

3. "Pixel spaces" – spaces of natural images (Gunnar Carlsson et al.).

Inflation $\Omega_\phi(K)$ of polyhedral complexes

Theorem: Suppose that K is a polyhedral complex on vertex set $V(K)$ and let P_K be the corresponding face poset ordered by the reversed inclusion. Suppose that $\phi : V(K) \to \mathbb{N}$ is the inflation function and let $\Omega_\phi(K)$ be the associated inflation complex. Assume that K is connected. Then,

$$\Omega_\phi(K) \simeq \bigvee_{F \in K} \Omega_{\phi|V(F)}(F) * \Delta((P^K)_{<F})$$